Space of Kähler metrics III—On the lower bound of the Calabi energy and geodesic distance

Xiuxiong Chen*
Department of Mathematics
University of Wisconsin, Madison

Contents

*The author is partially supported by NSF grant.

1 Introduction

Inspired by the beautiful work of Donaldson [15], the author initiated series of works [9] [8] aiming to understand geometric structure of the space of Kähler potentials and its application to interesting problems in Kähler geometry. The existence of $C^{1,1}$ geodesic is established in [9], while the best regularity result on geodesic is given by [12]. However, the approach taken in [12] is new (cf, Section 2 for further explanation.) and in many ways, the present work should be viewed as part III of this series. It consists of three inter-related parts:

1. First, we prove a folklore conjecture on the greatest lower bound of the Calabi energy in any Kähler class. This was known in the 1990s when Kähler metrics is assumed to be invariant under some maximal compact subgroup [23]. See ackwnoledgements for further remarks on this result.

2. Secondly, we give an upper/lower bound estimate of the K energy in terms of the geodesic distance and the Calabi energy. This is used to prove a theorem on convergence of Kähler metrics in holomorphic coordinates, with uniform bound on the Ricci curvature and the diameter. This kind of problems is difficult because the Kähler geometry is more or less extrinsic while the well known Cheeger-Gromov convergence theorem (with bound on curvature, diameters) is very intrinsic.

3. Thirdly, we set up a framework for the existence of geodesic rays when an asymptotic direction is given. In particular, if the initial geodesic ray is tamed by a bounded ambient geometry (c.f. Definition 3.2 and 3.3), then one can derive some relative $C^{1,1}$ estimates for other geodesic rays in the same direction. More in depth discussions on geodesic rays will be delayed to the beginning of Section 3. In a sequel of this paper, we will give more regularity estimates on geodesic rays.

1.1 Donaldson's Conjectures

According to Calabi [6], an extremal Kähler metric is charicterized as the critical points of the L^2 norm of the scalar curvature function in any given Kähler class. The extremal Kähler metric includes the more famous Kähler Einstein metric as a special case. In [15], Donaldson set out an ambitious program in attacking core problems of Kähler geometry via setting up formally a connection between geometric problems in the infinite dimesional space and the current interesting problems in Kähler geometry. In particular, he proposed three inter-related conjectures:

1. The space of Kähler potentials is uniquely connected by C^∞ geodesic segments;

2. The space of Kähler potentials is a metric space;

3. The non-existence of constant scalar curvature is equivalent to a geodesic ray where the K energy functional(c.f. eq. (2.7) for definition) decays at ∞.

In [9], following Donaldson's program, the author established the existence of $C^{1,1}$ geodesic by solving a Drichelet boundary value problem for a homogenous complex Monge Ampere equation. Consequently, the second conjecture of Donaldson is completely verified. Moreover, one important application is to show that Calabi's extremal Kähler metric (CextrK) is unique if the first Chern class is non-positive. The uniqueness problem is completely settled now: In algebraic manifold with discret automorphism group, it is proved by S. K. Donaldson [16]. T. Mabuchi [26] removes the assumption on the automorphism group while X. X. Chen-G. Tian [12] complete the proof for general Kähler class.

Chen-Tian [12] showed that the solution to disc version of geodesic problem is smooth except at most a codimension 2 set with respect to generic boundary data. For the convenience of readers, we will briefly describe the viewpoint of [12] in Section 2. In particular, the partial regularity theory established in [12] for solution of the disc version geodesic equation plays a crucial role in this paper (for obtaining a lower bound of the Calabi energy).

1.2 Yau-Tian-Donaldson conjecture

The Calabi conjecture on the existence of Kähler Einstein metrics has driven the subject for the second half of the last century. In late 1990s, S. T. Yau conjectured that the existence of Kähler Einstein metric in Fano manifolds is equivalent to some form of Stability of the underlying polarized Kähler class. According to G. Tian [32] and Donaldson [15], this equivalence relation should be extended to include the case of the constant scalar curvature (cscK) metric in a general Kähler class. In a foundational paper, G. Tian [32] introduced the notion of K Stability and in the same paper, he proved that the existence of KE metric implies weak K stability. More recently, in a fundamental paper [16], Donaldson proved that, in algebraic manifold with discrete automorphism group, the existence of cscK metric implies that the underlying Kähler class is Chow-Stable. In this paper, Donladson actually formulated a new version (but equivalent) of K stability in terms of weights of Hilbert points. In Kähler toric varieties, the existence of cscK metric implies the underlying Kähler class is Semi-K stable [18]. In [12] , Chen-Tian proved that the existence of a cscK metric implies that the K energy has a lower bound in this Kähler class. Following the work of Paul-Tian [27], this in turns implies the Semi-K stability of the underlying complex structure. After we announced our work [12], S. K. Donaldson[17] proved a similar lower bound in the algebraic settings.

1.3 On the existence of geodesic rays

The result of Donaldson on stability was extended by T.Mabuchi to the case of extremal Kähler metric with some modified notion of stability. However, for general Kähler classes, the usual notion of stability doesn't apply because the manifold can not be embedded in $\mathbb{C}P^N$ for some large $N \gg 1$. In [15], Donaldson envisioned that a geodesic segment or geodesic ray should play a similar role that a one parameter subgroup plays in a projective Kähler manifold. In the third conjecture of Donaldson's program, he defines a set of equivalence relationships:

1. There exists no constant scalar curvature metric in $(M, [\omega_0])$;
2. There exists a geodesic ray from some $\varphi_0 \in \mathcal{H}$ such that the K energy function is strictly decreasing as $t \to \infty$;
3. From any $\varphi \in \mathcal{H}$, there exists a geodesic ray initiated from φ such that the K energy function is strictly decreasing as $t \to \infty$.

The first step towards proving this conjecture is to establish an existence result of geodesic ray with respect to another given geodesic rays. According to Calabi-Chen[8], the infinite dimensional space $\mathcal{H}$ is a non-positively curved space. By the Triangle comparison theorem, we can show that there always exists a geodesic ray initiated from a given potential function in the direction of any given geodesic ray. However, the geodesic ray arisen this fashion acquired very little regularity and it is very hard to use in practice. As a first step in this direction, we prove

Theorem 1.1. *(cf. Theorem 3.8) If there exists a geodesic ray(c.f.Defi. 3.2) $\rho(t) : [0, \infty) \to \mathcal{H}$ which is tamed by a bounded ambient geometry, then for any Kähler potential $\varphi_0 \in \mathcal{H}$, there exists a relative $C^{1,1}$ geodesic ray $\varphi(t)$ initiated from φ_0. Moreover, this geodesic ray is parallel to the original geodesic ray.*

A geodesic ray tamed by a bounded ambient geometry is more or less "parallel" to the notion of special degeneration of complex structure in algebraic case. In Section 3, we will discuss in length, various issues related to stability (in terms of geodesic rays). It is expected that these notions are more or less equaivlent to the corresponding notions in the algebraic settings. We defer our discussions of this topic to the beginning of Section 3.

1.4 On the lower bound of geodesic distance and the collapsing of Kähler manifold

The famous work of Cheeger-Gromov states that the set of Riemannian metrics with the following three conditions:

1. uniform curvature bound,
2. diameter is bounded from above,

3. volume is bounded from below,

then this is compact under $C^{1,\alpha}$ diffeomphism for some $\alpha \in (0,1)$. If the third condition is dropped, then "collpasing" may occur (volume converges to 0). On the other hand, for any sequence of Kähler metrics in a fixed Kähler class, the volume is *a priori* fixed. With uniform control of the curvature and diameter from above, it will converge by subsequence to some Kähler metric with perhaps a different complex structure. In general, we don't know what additional geometrical condition is needed to ensure that the limit complex structure is the same with the original complex structure. In fact, such a sequence might collapse in some Zariski open subset of the original Kähler manifold (i.e., the volume form vanishes in this subset) while the subsequence of Kähler metrics converges as Riemmanian metrics up to diffeomorphism (cf. [30]). In the discussion below, we will refer this phenomenon as "Kähler collapsing. "

One intriguing and challenging question is: when this "Kähler collpasing" occurs, does the geodesic distance (in the space of Kähler metrics) necessary diverge to ∞? [1] This in turns leads to another question: how do we estimate the lower bound of the geodesic distance? For instance, if the diameter of a sequence of Kähler metrics in a given Kähler class diverges to ∞, does the geodesic distance of this sequence of Kähler metrics also diverge to ∞?

We first prove a theorem which links the K energy, the Calabi energy and the geodesic diameter together. The author believes that this theorem is very interesting in its own right.

Theorem 1.2. *Let φ_0, φ_1 are two arbitrary Kähler potentials in the same Kähler class. Then, the following inequality holds*

$$\mathbf{E}(\varphi_1) - d(\varphi_0, \varphi_1) \cdot \sqrt{Ca(\varphi_1)} \leq \mathbf{E}(\varphi_0). \tag{1.1}$$

Here $d(\varphi_0, \varphi_1)$ is the geodesic distance in the space of Kähler potentials.

In other words: if geodesic distance and Calabi energy is bounded, so is the upper bound of the K energy. This is quite surprising since we don't know how to control the K energy, even after one assumes the uniform bound of the Riemannian curvature. On the other hand, fixing φ_1 and let φ_0 change, this formula gives a lower bound estimate of the K energy in terms of geodesic distance as well. Clearly, this inequality is a natural generalization of the theorem [12] that the K energy has a lower bound if there is a cscK metric. In fact, we conjecture that, in a fixed Kähler class, if the infimum of the Calabi energy reaches 0, then the K energy must have a lower bound.

An immediate corollary is:

[1] For a sequence of metrics mentioned above which does not converge in the original complex structure, one is expected to prove, via implicite function theory, that the geodesic distance (to some fixed Kähler metrics) must diverges to ∞.

Corollary 1.3. *Let φ be a Kähler potential such that its Calabi energy is bounded. If $|\varphi|_\infty$ is bounded, then its K energy is bounded from above.*

We say the K energy functional is "proper" if it is bounded from below by certain norm function which will be introduced in Section 2. We say the K energy functional is "quasi-proper" if the K energy functional is bounded below by its highest order leading term (cf. Section 2). In Kähler Einstein manifold, the K energy functional is always proper[32]. In a general Kähler manifold, Tian conjectured that the cscK metric exists if and only if the K energy functional is proper. When the first Chern class is semi-negative, there is a sufficient condition that the K energy functional in that Kähler class is either proper or quasi proper [2].

Now we are ready to answer the question about "Kähler collpasing":

Theorem 1.4. *(No "Kähler collapsing") Let $(M, [\omega])$ be a polarized Kähler manifold where the K energy functional is either proper or quasi-proper. Let $\mathcal{S}$ be a set of Kähler metrics in $[\omega]$ with uniform Ricci curvature bound from below and diameter bound from above [3]. If this set of Kähler metrics lies in a bounded geodesic ball in the space of Kähler metrics, and if we assume Ricci also has an upper bound, then all metrics in $\mathcal{S}$ are uniformly equivalent to each other in $C^{1,\alpha}(M)$ topology for any $\alpha \in (0,1)$. In particular, "Kähler collapsing" will not occur.*

Note that the geodesic distance appears to be a very weak notion. The bound on Ricci curvature is much weaker than the conditions stated in Cheeger-Gromov's theorem. However, the combination of the two conditions seems to be very powerful.

In a subsequent paper, we will drop the assumption that the Ricci is bounded from above. The assumption that the K energy functional is either proper or quasi-proper in $(M, [\omega])$ is just technical. We hope that this will be removed in a subsequent work.

1.5 On the lower bound of the Calabi energy

It is well known that the Calabi energy is locally convex near an extremal Kähler metric. It is a very interesting and difficult question if the Calabi energy in the Kähler class is bounded below by the energy of the extremal metric. According to Calabi[6], an extremal Kähler metric automatically exhibits the maximal

[2]For instance, in complex dimension 2, if

$$\frac{[\omega]\cdot[-C_1(M)]}{[\omega]^{[2]}}(-C_1(M)) - [\omega] > 0$$

then the K energy is quasi-proper [10]. For higher dimension Kähler manifolds, readers are referred to Song-WeinKove[3] .

[3]The diameter bound can be replaced by a bound on Sobleve constant.

symmetry possible allowed by the underlying complex structure. In the 1990s, A. Hwang [23] proved that the Calabi energy of the invariant Kähler metrics (maximal possible symmetric...) is bounded below by the absolute value of the Futaki invariant (evaluated at the Canonical extremal vector field). If there is an extremal Kähler metric in this class, the absolute value of the Futaki invariant is precisely the Calabi energy of the extremal Kähler metrics. Hwang's proof uses strongly the bi-invariant metric in the Lie algebra of gradient holomorphic vector fields where the symmetric property is the key to define this positive definite invariant metric. In 1980s, it is conjectured that the same low bound holds for all metrics in the same Kähler class. There are many attempts to generalize this to all Kähler metrics, and this problem has proved to be very difficult indeed.

Aside from this Folklore conjecture on the Calabi energy, there are other important motivations to study the lower bound of the Calabi energy *a priori*, for instance, the issue related to stability and degeneration of Kähler manifolds. For our strategy to work, the main technical obstacle has been the insufficient regularity of the $C^{1,1}$ geodesic. However, the partial regularity theory established in [12] for solution of the disc version geodesic equation plays crucial role here. In particular, it is a very powerful fact that the restriction of the K energy of this family of Kähler potential over disc is subharmonic. We are able to use this fact to establish a lower bound for the Calabi energy in terms of any effective destabilizing geodesic ray (cf. Defi.3.12). In particular, we prove this folklore conjecture about the lower bound of the Calabi energy in each Kähler class.

Theorem 1.5. *Let $\mathcal{K}$ be the Lie algebra of complex gradient holomophic vector field. Then for any Kähler metric ω_g in $[\omega]$, we have*

$$Ca(\omega_g) \geq \mathcal{F}_{\mathcal{X}_c}([\omega]).$$

where $\mathcal{X}_c$ is the a priori extremal vector field in $(M, [\omega])$ and $\mathcal{F}$ is the Futaki invariant. The equality holds when g is an extremal Kähler metric.

More generally, we have

Theorem 1.6. *Suppose $\rho(t) : [0, \infty) \to \mathcal{H}$ is an effective destabilized geodesic ray in $\mathcal{H}$, then*

$$\inf_{\varphi \in \mathcal{H}} \int_M (R(\varphi) - \underline{R})^2 \omega_\varphi^n \geq \sup_\rho \yen(\rho)^2, \tag{1.2}$$

where the sup in the right hand side runs over all possible destabilized geodesic rays. The definition of an effective destabilized geodesic ray and $\yen$ invariant are given in Defi.3.12 and Defi. 3.9 respectively.

Definition 1.7. *Let $(M, [\omega_0], J_0)$ be a triple Kähler structure. Another triple Kähler structure $(M', [\omega'], J')$ lies in the closure of the diffeomorphism orbit*

of $(M, [\omega_0], J_0)$ if there exists a sequence of Kähler forms $\{\omega_{\varphi_m}, m \in \mathbb{N}\} \subset [\omega]$ and a sequence of diffeomorphism $\{f_m \in Diff(M), m \in \mathbb{N}\}$ such that $(M, f_m^* \omega_{\varphi_m}, f_m^* J_0)$ converges to (M', ω', J') in $C^{1,\alpha}$ topology for some $\alpha \in (0, 1)$.

Definition 1.8. *Let $(M, [\omega_0], J_0)$ be a triple Kähler structure. Suppose that $(M', [\omega'], J')$ is another triple Kähler structure which lies in the closure of the diffeomorphism orbit of $(M, [\omega_0], J_0)$. $(M', [\omega'], J')$ is called destabilizer of the original triple Kähler structure $(M, [\omega_0], J_0)$ if there exists an effective destabilized geodesic ray $\varphi(t)$ in $(M, [\omega_0], J_0)$ such that there is a subsequence of $\omega_{\varphi(t)} (t \to \infty)$ which converges to a metric in $(M', [\omega'], J')$ up to diffiomorephism.*

Now we can extend Theorem 1.5 to a more general setting:

Theorem 1.9. *Let $(M, [\omega_0], J_0)$ be a triple Kähler structure. The following inequality hold*

$$\inf_{g \in \mathcal{H}} Ca(\omega_g) \geq \sup_{(M', [\omega'], J')} \inf_{(X,X)=1, X \in \mathcal{K}(J')} \frac{(X, \mathcal{X}_c)^2}{(X, X)}.$$

Here the inner product is the Futaki-Mabuchi inner product for the Lie algebra $\mathcal{K}(J')$ of the Maximum compact subgroup $K(J')$ of $Aut(M', J')$. The supremum runs over all possible Kähler triple structures $(M', [\omega'], J')$ in the closure of the diffeomorphism orbit of $(M, [\omega_0], J_0)$ which destabilized $(M, [\omega_0], J_0)$.

One should be able to define a weak notion of destablizing Kähler triple later, while the inequality in the preceding theorem still hold.

Definition 1.10. *Suppose the Kähler triple (M, ω, J) satisfies the following inequality*

$$\mathcal{F}_{\mathcal{X}_c}([\omega_0]) \geq \sup_{(M', [\omega'], J')} \mathcal{F}_{\mathcal{X}_c}([\omega']), \tag{1.3}$$

where $(M', [\omega'], J')$ are all Kähler triples in its closure of diffemorphisms. Then we call $(M, [\omega], J)$ stable in the sense of differential geometry.

An immediate interesting/challenging question is: what is the relation of stablity in the sense of differential geometry with other notions of stability such as K stablity? In algebraic manifold, the notion of K stability shall be stronger than this one.

In light of these theorems, one expects that there is a deep relation between geodesic rays, test configurations and their respective role in defining stability. We then propose some notions of stability in terms of geodesic rays which might be viewed as a natural extension of what is given in [15]. Moreover, the relation between geodesic stability and K stability should be also an interesting topic to explore in near future. More extensive discussions on this topic will be delayed to Section 3.

Organization: In Section 2, we give a brief outline of known results in the space of Kähler potentials. In Section 3, we prove that the existence of geodesic ray with respect to some "nice" geodesic ray. In Section 4, we give a greatest lower bound estimate for the Calabi energy. In Section 5, we give a lower bound estimate of the geodesic distance and rule out the possibility of "Kähler collapsing" in bounded geodesic balls in $\mathcal{H}$.

Acknowledgment: The strategy of obtaining a lower bound of the Calabi energy through geodesic ray has been discussed with S. K. Donaldson in 1997-98, on and off since then. The author wants to thank Professor Donaldson for kindly sharing his insight on this matter. Readers are encouraged to compare the results on lower bound of Calabi energy to [18] (In particular, Theorem 1.6.).

Thanks also goes to Professor G. Tian, our joint work [12] really provided technical support to Theorem 1.2. Thanks also to Professor Calabi and Professor J. P. Bourguinon for their continuous support in my research in last few years. My student Yudong Tang carefully read through an earlier version of this paper and I want to thank him for his help.

2 Brief outline of geometry in the Space of Kähler potentials.

2.1 Quick introduction of Kähler geometry

Let ω be a fixed Kähler metric on M. In a holomorphic coordinate, ω can be expressed as

$$\omega = g_{\alpha\bar\beta}\,\frac{\sqrt{-1}}{2}\,dw^\alpha \wedge d\,w^{\bar\beta}.$$

The Ricci curvature can be conveniently expressed as

$$R_{\alpha\bar\beta} = -\frac{\partial^2 \log\det\left(g_{i\bar j}\right)}{\partial w^\alpha \partial w^{\bar\beta}}.$$

The scalar curvature can be defined as

$$R = -g^{\alpha\bar\beta}\,\frac{\partial^2 \log\det\left(g_{i\bar j}\right)}{\partial w^\alpha \partial w^{\bar\beta}}.$$

The so called Calabi energy is

$$Ca(\omega) = \int_M (R(\omega) - \bar R)^2 \omega^n. \tag{2.1}$$

Here $\bar R$ is the average scalar curvature value for all metric in the Kähler class. According to Calabi [6] [7], a Kähler metric is called extremal if the complex gradient vector field

$$\mathcal{X}_c = g^{\alpha\bar\beta}\frac{\partial R}{\partial w^{\bar\beta}}\frac{\partial}{\partial w^\alpha} \tag{2.2}$$

is a holomorphic vector field. According to [21], the extremal vector field $\mathcal{X}_c$ is *a priori* determined in each Kähler class, up to holomorphic conjugation.

If X is a holomorphic vector field, then for any Kähler potential φ we can define θ_X up to some additive constants by

$$L_X \omega_\varphi = \sqrt{-1}\partial\bar{\partial}\theta_X. \tag{2.3}$$

Then, the well known Calabi-Futaki invariant [20] [7] is

$$\mathcal{F}_X([\omega]) = \int_M \theta_X \cdot (\underline{R} - R(\varphi))\, \omega_\varphi^n. \tag{2.4}$$

Note that this is a Lie algebra character which depends on the Kähler class only.

2.2 Weil-Peterson type metric by Mabuchi

It follows from the Hodge theory that the space of Kähler metrics with Kähler class $[\omega]$ can be identified with the space of Kähler potentials

$$\mathcal{H} = \{\varphi \mid \omega_\varphi = \omega + \bar{\partial}\partial\varphi > 0, \text{ on } M\}/\sim,$$

where $\varphi_1 \sim \varphi_2$ if and only if $\varphi_1 = \varphi_2 + c$ for some constant c. A tangent vector in $T_\varphi\mathcal{H}$ is just a function ψ such that

$$\int_M \psi\omega_\varphi^n = 0.$$

Its norm in the L^2-metric on $\mathcal{H}$ is given by (cf. [25])

$$\|\psi\|_\varphi^2 = \int_M \psi^2\, \omega_\varphi^n.$$

It was subsequently defined similarly in [31] and [15]. In all three papers, [25][31] and [15], the authors defined this Weil-Peterson type metric from various points of view and proved formally that this infinite dimensional space has non-positive curvature. Using this definition, we can define a distance function in $\mathcal{H}$: For any two Kähler potentials $\varphi_0, \varphi_1 \in \mathcal{H}$, let $d(\varphi_0, \varphi_1)$ be the infimum of the length of all possible curves in $\mathcal{H}$ which connects φ_0 with φ_1.

A straightforward computation shows that a geodesic path $\varphi : [0,1] \to \mathcal{H}$ of this L^2 metric must satisfies the following equation

$$\varphi''(t) - {g_\varphi}^{\alpha\bar{\beta}} \frac{\partial^2\varphi}{\partial t \partial w^\alpha} \frac{\partial^2\varphi}{\partial t \partial w^{\bar{\beta}}} = 0.$$

where

$$g_{\varphi,\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} + \frac{\partial^2\varphi}{\partial w^\alpha \partial w^{\bar{\beta}}}.$$

According to S. Semmes [31], this path $\{\varphi(t)\}$ satisfies the geodesic equation if and only if the function ϕ on $[0,1]\times S^1\times M$ satisfies the homogeneous complex Monge-Ampere equation

$$(\pi_2^*\omega + \partial\overline{\partial}\phi)^{n+1} \quad = \quad 0, \qquad \text{on } \Sigma\times M, \tag{2.5}$$

where $\Sigma = [0,1]\times S^1$ and $\pi_2 : \Sigma\times M \mapsto M$ is the projection. In fact, one can consider (2.5) over a general Riemann surface Σ with boundary condition $\phi = \phi_0$ along $\partial\Sigma$, where ϕ_0 is a smooth function on $\partial\Sigma\times M$ such that $\phi_0(z,\cdot)\in\mathcal{H}$ for each $z\in\partial\Sigma$.[4] It also has geometric meaning. The equation (2.5) can be regarded as the infinite dimensional version of the WZW equation for maps from Σ into $\mathcal{H}$ (cf. [15]).[5]

Next we introduce three well known functionals in $\mathcal{H}$ here. First, the so called I functional is defined as

$$\frac{dI(\varphi(t))}{d\,t} = \int_M \frac{\partial\,\varphi}{\partial\,t}\omega^n_{\varphi(t)}, \qquad \varphi(t)\in\mathcal{H}.$$

The advantage of the and I functional is that it is a constant along geodesic. One can write down an explicit formula for I functional

$$I(\varphi) = \int_M \varphi\,\omega^n - \sum_{k=0}^{n-1}\frac{n-k}{n+1}\,\partial\varphi\wedge\bar\partial\varphi\wedge\omega^k\wedge\omega_\varphi^{n-k-1}. \tag{2.6}$$

We write down a detailed proof for $I(\varphi)$ in Section 5 (prop **??**).

Secondly, the so called J funtional is defined as

$$J(\varphi) = \int_M \varphi\left(\omega^n - \omega_\varphi^n\right) = \int_M \partial\varphi\wedge\bar\partial\varphi\wedge\left(\sum_{k=0}^{n-1}\omega^k\wedge\omega_\varphi^{n-k-1}\right) > 0.$$

Finally, the K energy functional (introduced by T. Mabuchi) is defined as a closed form $d\,\mathbf{E}$. Namely, for any $\psi\in T_\varphi\mathcal{H}$, we have

$$(d\,\mathbf{E},\psi)_\varphi = \int_M \psi\cdot(\underline{R} - R(\varphi))\,\omega_\varphi^n. \tag{2.7}$$

Note that for any holomorphic vector field, we have

$$\mathcal{F}_X([\omega]) = (d\,\mathbf{E},\theta_X)_\varphi.$$

The K energy functional is called proper in $(M,[\omega])$ if there exists a small constant $\delta>0$ a constant C such that

$$\mathbf{E}(\varphi)\geq J(\varphi)^\delta - C.$$

[4]We often regard ϕ_0 as a smooth map from $\partial\Sigma$ into $\mathcal{H}$.
[5]The original WZW equation is for maps from a Riemann surface into a Lie group.

The K energy functional is called proper in $(M, [\omega])$ if there exists a small constant $\delta > 0$ and a constant C such that

$$\mathbf{E}(\varphi) \geq \delta \int_M \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n - C \tag{2.8}$$

2.3 The new approach in Chen-Tian's paper [12]

The lack of sufficient regularity in Chen's solution to geodesic equation [9] is the obstruction to proving deeper results in general cases by using geodesics approaches. We should point out that complex Monge-Ampere equations have been studied extensively (cf. [5], [24], [2] etc.). However, regularity for solutions of homogeneous complex Monge-Ampere equations beyond $C^{1,1}$ has been missing. Indeed, there are examples in which some solutions are only $C^{1,1}$.

The best regularity result about geodesic segment is due to Chen-Tian [12] where they showed that the solution to the disc version of the geodesic problem is smooth except at most a codimension 2 set with respect to generic boundary data. Although the $C^{1,1}$ bound derived in [9] plays a crucial role, Chen-Tian takes a new viewpoint towards geodesic equation. The main theorem in [12] is

Theorem 2.1. *[12] Suppose that Σ is a unit disc. For any $C^{k,\alpha}$ map $\phi_0 : \partial\Sigma \to \mathcal{H}$ ($k \geq 2$, $0 < \alpha < 1$) and for any $\epsilon > 0$, there exists a $\phi_\epsilon : \partial\Sigma \to \mathcal{H}$ in the ϵ-neighborhood of ϕ_0 in $C^{k,\alpha}(\Sigma \times M)$-norm, such that (2.5) has an almost smooth solution with boundary value ϕ_ϵ.*

An almost smooth solution of eq. 2.5 has a uniform $C^{1,1}$ bound and smooth almost everywhere. A detailed explanation (including definitions) can be found in [12]. However, the importance of this theorm lies in the following

Theorem 2.2. *[12] Suppose that ϕ is a partially smooth solution to (2.5). For every point $z \in \Sigma$,, let $\mathbf{E}(z)$ be the K-energy (or modified K energy) evaluated on $\phi(z, \cdot) \in \overline{\mathcal{H}}$. Then $\mathbf{E}$ is a bounded subharmonic function on Σ in the sense of distributions, moreover, we have the following*

$$\int_{\mathcal{R}_\phi} |\mathcal{D}\frac{\partial\phi}{\partial z}|^2_{\omega_{\phi(z,\cdot)}} \omega_{\phi(z,\cdot)}{}^n \, d\,z d\,\bar{z} \leq \int_{\partial\Sigma} \frac{\partial \mathbf{E}}{\partial \mathbf{n}}|_{\partial\Sigma} ds,$$

where ds is the length element of $\partial\Sigma$ and for any smooth function θ, $\mathcal{D}\theta$ denotes the (2,0)-part of θ's Hessian with respect to the metric $\omega_{\phi(z,\cdot)}$. The equality holds if ϕ is almost smooth.

3 On the existence of geodesic ray

3.1 Definitions and main results

As suggested in [15], one should view the geodesic rays as effective substitute of a one parameter family of subgroup acting on projective Kähler manifolds. It

is natural to compare geodesic rays to the test configurations $\pi : \mathcal{X} \to \triangle$ such that all fibre Kähler manifolds $\pi^{-1}(t)$ are bi-holomorphic to each other except when $t = 0$. The central fibre usually carries a different complex structure with singularities. The generic case is so called "Normal cross singularity" and the special case is when the central fibre is either smooth or has singular local codimenison 4 or higher. However, by blowing up a few points in the central fibre if necessary, it might be possible to make the total space smooth or have some form of bounded geometry. For any test configuration, it might be possible to prove that there is always a relatively $C^{1,1}$ geodesic ray which is asymtotically closed to the test configuration near the central fiber. If the central fibre is smooth or smooth except a subvariety of condimension 4, then the geodesic ray is smooth generically except perhaps a singular locus of codimenion two or higher.

Motivated from the study of test configuration in algebraic setting, in this section, we restrict our attentions to the case of "nice" geodesic rays $\omega_{\rho(t)}(t \in [0,\infty))$ which satisifies the following conditions:

1. The non-compact family $\left(M, \omega_{\rho(t)}\right)$ can be compactified in some sense;
2. The limit of $(M, \omega_{\rho(t)})$ as $t \to \infty$ under suitable topology is smooth in the "compactiftication" or has mild singularities (codimension 4 and higher).

The most special case of geodesic rays are those arising from a fixed gradient complex holomorphic vector field. In this case, the curvature of $(M, \omega_{\rho(t)})$ is uniformly bounded and the injectivity radius is uniformly bounded from below.

Consider $\pi_2 : ([0,\infty) \times S^1) \times M \to M$ as natural projection map.

Definition 3.1. *A path $\rho(t) : [0,\infty) \to \mathcal{H}$ is called strictly convex if $\pi_2^*\omega_0 + i\partial\bar{\partial}\rho$ defines a Kähler metric in $([0,\infty) \times S^1) \times M$.*

Definition 3.2. *A geodesic ray $\rho(t)(t \in [0,\infty))$ is called special if it is one of the following types:*

1. **effective** *if the Calabi energy of $\omega_{\rho(t)}$ in M is dominated by $\frac{\epsilon}{t^2}$ for any $\epsilon > 0$ as $t \to \infty$.*
2. **normal** *if the curvature of $\omega_{\rho(t)}$ in M is uniformly bounded for $t \in [0,\infty)$.*
3. **bounded geometry** *if $(M, \omega_{\rho(t)})(t \in [0,\infty))$ has uniform bound on curvature and a uniform positive lower bound of injective radius.*

Definition 3.3. Tamed by a bounded ambient geometry *A Kähler metric $h = \pi_2^*\omega_0 + i\partial\bar{\partial}\bar{\rho}$ in $([0,\infty) \times S^1) \times M$ is said to have bounded ambient geometry if*

1. *it has a uniform bound on its curvature;*

2. $([0,T]\times S^1\times M, h)$ *has a uniform lower bound on injectivity radius and the bound is independent of* $T\to\infty$;

3. *The vector length* $|\frac{\partial}{\partial t}|_h$ *has a uniform upper bound.*

A geodesic ray $(M,\omega_{\rho(t)})$ *is called tamed by this ambient metric* h *if there is a uniform bound of the relative potential* $\rho-\bar\rho$, *or if there is a uniform constant* C *such that* [6]

1. $\max_t\ |n+1+\triangle_h(\rho-\bar\rho)|\le C;$

2. $\max_t\ |\dfrac{\partial(\rho-\bar\rho)}{\partial t}|_h\le C;$

Remark 3.4. *A geodesic ray, tamed by a bounded ambient geometry, corresponds to the special degeneration of the complex structure in the algebraic setting. In the future, we should broad our definition of bounded ambient geometry to include the following situations:*

1. *The upper bound of the curvature of the ambient Kähler metric might not be uniform;*

2. *The injectivity radius may have a lower bound which depends on the distance to some singular subvariety of higher codimension as well as on* t;

3. *The restriction of the ambient Kähler metric* h *in* $\{t_i\}\times S^1\times M$ *may have some finite geodesic distance to* $(M,\omega_{\rho(t_i)})$ *while the later has certain geometric bounds (such as the Calabi energy or Sobelev constant, cf. Theorem 1.4).*

Of course, the regularity of geodesic may be weaked a bit as well.

Remark 3.5. *Using Cauchy-Kowalevaski's classical theorem, Arezzo-Tian [1] proved that, a special degeneration of a complex structure when the central fibre is analytic, is asymptotically equivalent to a geodesic ray near the central fibre.*

Example 3.6. *Suppose that* X *is a gradient holomorphic vector field and let* ω_0 *be a Kähler form invariant under* $Im(X)$. *Let* $\sigma(t), t\in[0,\infty)$ *be the automorphism group generated by* X. *Set*

$$\omega_{\rho(t)}=\sigma_t^*\omega_0.$$

A straightforward calculation shows that $\rho(t)(t\in(-\infty,\infty)$ *is a geodesic line. Let* $\sigma=\sigma_1$ *and let* $g_1=\sigma^*g_0$ *and* g_0 *be the two Kähler metrics corresponding to* ω_0 *and* $\sigma^*\omega_0$. *Note that*

$$z\frac{\partial}{\partial z}+X$$

[6]It is possible to only assume these two inequsalities holds for a sequence of $t_i\to\infty$.

induces a $\mathbb{C}^*$ action $\bar{\sigma}$ on $\Delta\times M$ which coincides with σ in the manifold direction and the multiplicity action on Δ direction. Let $z_0 = 1$ and $z_k = \bar{\sigma}^k z_0 \to 0$. Set

$$M_{l,k} = \{\frac{1}{2^l} \leq |z| \leq \frac{1}{2^k}\} \times M, \qquad \forall\ l,k \in \mathbb{N}.$$

Then

$$M_{0,\infty} = (\Delta \setminus \{0\}) \times M.$$

It is easy to see that there is a smooth S^1 invariant Kähler metric $\bar{h}$ in $M_{0,1}$ such that

1. $\bar{h}\ |_{|z|=0}= g_0$ *and* $\bar{h}\ |_{|z|=1}= g_1$.
2. h *and* $\bar{\sigma}^*\bar{h}$ *gives rise a smooth metric in* $M_{0,2}$.

Using $\bar{h}$, we can define a Kähler h in $(\Delta \setminus \{0\}) \times M$ simply by

$$h(z,\cdot) = \bar{\sigma}^{k*}\bar{h}, \qquad \forall\ (z,\cdot) \in M_{k,k+1}.$$

By definition, h is a smooth metric in $(\Delta\setminus\{0\})\times M$ which has bounded curvature and uniform positive lower bound on injectivity radius.

In fact, any normal geodesic ray is expected to be tamed by some bounded ambient geometry, at least when it has bounded geometry.

Definition 3.7. *For any two geodesic rays $\rho_1(t), \rho_2(t) : [0,\infty) \to \mathcal{H}$, they are called parallel if there exists two constants C such that*

$$\sup_{t\in[0,\infty)} (\rho_1(t) - \rho_2(t)) \leq C.$$

Theorem 3.8. *If there exists a geodesic ray $\rho(t) : [0,\infty) \to \mathcal{H}$ which is tamed by an ambient geometry, then for any Kähler potential $\varphi_0 \in \mathcal{H}$, there exists a relative $C^{1,1}$ geodesic ray $\varphi(t)$ initiated from φ_0 and parallel to $\rho(t)$ such that*

$$\sup_{t\in[0,\infty)} \mid \rho(t) - \varphi(t) \mid \leq C.$$

Denote $\rho - \bar{\rho}$ is the relative Kähler potential of the given geodesic ray with respect to the ambient metric h. If $\rho - \bar{\rho}$ is uniformly C^2 bounded in manifold direction and if it is C^1 bound in t direction with respect to the ambient metric h, then there exists two uniform constants $,\lambda, C$ such that

$$0 \leq n + 1 + \tilde{\triangle}(\varphi(t,x) - \rho(t,x)) \leq C \exp \lambda(\rho(t,x) - \bar{\rho}(t,x)).$$

Here $\tilde{\triangle}$ is taken with respect to the ambient Kähler metric h. In particular, when[7]

$$\rho(t,x) - \bar{\rho}(t,x)$$

is uniformly bounded, the resulting geodesic ray has a uniformly $C^{1,1}$ bound in terms of ambient metric h. The constant λ, C depends on h.

[7]This is the case when the geodesic ray is given by one parameter holomorphic transfermation (c.f. ex. 3.5).

In [29], Phong and Jacob approximate the $C^{1,1}$ geodesic segment (established in [9]) in algebraic manifold via finite dimensional approach. In light of the preceeding theorem, it will be nice to approach this relative $C^{1,1}$ geodesic ray via finite dimensional approaches too.

Definition 3.9. *For every geodesic ray $\rho(t)(t \in [0,\infty)$, we can define an invariant as*

$$\yen(\rho) = \lim_{t\to\infty} \int_M \frac{\partial \rho(t)}{\partial t}(\underline{R} - R(\rho(t)))\omega^n_{\rho(t)}. \tag{3.1}$$

Remark 3.10. *For a smooth geodesic ray, the K energy is convex and the above invariant is well defined.*

In case the geodesic ray arises from a one parameter holomorphic transfermation, the integrand in equation 3.1 is just the usual Calabi-Futaki invariant. This invariant shall be compared to the genearlized Futaki invariant defined by Ding and Tian on Fano varieties.

A natural question is: If two geodesic rays are parallel to each other, are their $\yen$ invariants the same? The answer is partially"yes":

Proposition 3.11. *If one of the geodesic rays has bounded ambient geometry, then any other geodesic ray parallel to it must have same $\yen$ invariant.*

Definition 3.12. *A geodesic ray $\rho(t) : [0,\infty) \to \mathcal{H}$ is called stable (resp; semistable) if $\yen(\rho) > 0$ (resp: ≥ 0). It is called a de-stabalizer for $\mathcal{H}$ if $\yen(\rho) < 0$ and it is called an effective de-stablizer if in additional*

$$\limsup_{t\to\infty} \; t^2 \cdot \int_M (R(\rho(t)) - \underline{R})^2 \omega^n_{\rho(t)} = 0.$$

Following the approach in algebraic case, we define (cf. [15]):

Definition 3.13. *A Kähler manifold is called (effectively) geodesicly stable if there is no (effective) de-stablizing geodesic ray. It is called weakly geodesicly stable if the invariant $\yen$ is always non-negative for every geodesic ray.*

One of the main theorems is:

Theorem 3.14. *Suppose $\rho(t) : [0,\infty) \to \mathcal{H}$ is an effective de-stablized geodesic ray in $\mathcal{H}$, then*

$$\int_M (R(\varphi) - \underline{R})^2 \omega^n_\varphi \geq \yen(\rho)^2, \qquad \forall\, \varphi \in \mathcal{H}.$$

In fact, we have

$$\inf_{\varphi\in\mathcal{H}} \int_M (R(\varphi) - \underline{R})^2 \omega^n_\varphi \geq \sup_\rho \yen(\rho)^2, \tag{3.2}$$

where the sup in the right hand side of 3.2 runs over all possible effective destablized geodesic rays.

As a corollary, we have the following important consequence

Corollary 3.15. *If there is a Kähler metric of constant scalar curvature, then it is weakly effectively geodesic stable.*

One can generalized these results to the case of extremal Kähler metric with non-constant scalar curvature.

Definition 3.9a *Suppose $\mathcal{X}_c$ is the canonical extremal vector field in $(M, [\omega])$ (cf. eq. 2.2) and $\theta(\mathcal{X}_c)$ is defined as equation 2.3. For every geodesic ray $\rho(t)(t \in [0, \infty))$, we can define an invariant as*

$$\tilde{¥}(\rho) = \lim_{t \to \infty} \int_M \frac{\partial \rho(t)}{\partial t} (\underline{R} - R(\rho(t) - \theta(\mathcal{X}_c)))\omega^n_{\rho(t)}. \tag{3.3}$$

A geodesic ray $\rho(t) : [0, \infty) \to \mathcal{H}$ is called stable (resp; semi-stable) if $\tilde{¥}(\rho) > 0$(resp: ≥ 0). It is called a destabilizer for $\mathcal{H}$ if $\tilde{¥}(\rho) < 0$ and effective destablizer if in addition

$$\limsup_{t \to \infty} \ t^2 \cdot \int_M (R(\rho(t)) - \underline{R} - \theta(\mathcal{X}_c))^2 \omega^n_{\rho(t)} = 0.$$

With essentially same proof, we have

Theorem 3.2a *Suppose $\rho(t) : [0, \infty) \to \mathcal{H}$ is an effecitvely destabilizing geodesic ray in $\mathcal{H}$, then*

$$\int_M (R(\varphi) - \underline{R} - \theta(\mathcal{X}_c))^2 \omega^n_\varphi \geq \tilde{¥}(\rho)^2, \qquad \forall\, \varphi \in \mathcal{H}.$$

In fact, we have

$$\inf_{\varphi \in \mathcal{H}} \int_M (R(\varphi) - \underline{R} - \theta(\mathcal{X}_c))^2 \omega^n_\varphi \geq \sup_\rho \tilde{¥}(\rho)^2, \tag{3.4}$$

where the sup in the right hand side runs over all possible effectively destabilizing geodesic rays. Moreover, the underlying manifold is weakly geodesic stable if there exists an extremal Kähler metric in the Kähler class.

3.2 Proof of Theorem 3.8

In this subsection, we will give a proof of the existence of a geodesic ray when the initial geodesic ray has bounded ambient geometry. One of the main challenges here has been searching for the right condition for the existence of a parallel geodesic ray with regularity beyond the L^2 topology on the Kähler potential. Following the main steps in [9] under current circumstance: for any given Kähler potential φ_0, we can pick a sequence of Kähler metrics $\rho(t_i)(i \in \mathbb{N})$ along the given geodesic ray, and connects φ_0 to $\rho(t_i)(i \in \mathbb{N})$ via the unique $C^{1,1}$ geodesic segment established in [9]. This way, we obtain a sequence of $C^{1,1}$ geodesic

ray and hope to take a limit as $t_i \to \infty$. The main difficult is to obtain some uniform $C^{1,1}$ bound which allows us to take a limit as $t_i \to \infty$. However, such an approach runs into a serious problem as we shall explain now: first, there is no absolute C^0 estimate which is crucial to the Yau's calculation of the second derivatives. Secondly, when we do the blowing up estimate, the compactness of the underlying Kähler manifold becomes crucial. Thirdly, in deriving boundary estimate as in [9], we need the assumption that the restriction of Käher metric in $\partial\Sigma \times M$ has a uniform positive lower bound with respect to some fixed metric. This is clearly not available since the sequence of metrics along a geodesic ray are expected to either diverge or converge to a metric in different complex structure. In a typical senario, this sequence of metrics will degenerate along generic points in the Kähler manifold and will blowup along some divisor. To overcome this difficulty, we use this bounded ambient metric from the initial geodesic ray to obtain some control of C^0 bound on the modified potentials. In order to derive a C^2 estimate in terms of this weak C^0 estimate, we need to exploit the structure of degenerated Monge-Ampere equation more closely. In particular, if the modified potential doesn't have a uniform C^0 bound, we need to re-design the blowing up procedure in [9] to obtain a growth control in the $C^{1,1}$ bound on the modified potential. We believe that such a technique may be applicable to some other interesting cases.

3.2.1 Setup of problem

Let us first set up some notations. Let $T \gg 1$ be a generic large positive number. Let $\Sigma_T = [0, T] \times S^1$. In the $(n+1)$ dimensional Kähler manifold $\Sigma_T \times M$, we want to solve the Dirichelet problem for HCMA equation 2.5 where the boundary data is invariant in the circle S^1 direction. As in [9], for any T and for any smooth boundary data, we can obtain a unique $C^{1,1}$ solution $\phi(t)$ such that it solves HCMA equation 2.5. In other words, we have

$$\left(\pi_2^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi\right)^{n+1} = 0, \tag{3.5}$$

where

$$\phi(0) = \varphi_0, \qquad \text{and} \qquad \phi(T) = \rho(T). \tag{3.6}$$

Here we assume that $\rho : [0, \infty) \to \infty$ is a smooth geodesic ray. Obviously, the $C^{1,1}$ estimate depends on T and may blow up as $T \to \infty$. In fact, this $C^{1,1}$ estimate must blow up if it represents a geodesic ray. This creates a serious problem for the existence of geodesic rays. Our strategy is the following: let $\{n_k \in \mathbb{N}\}$ be a sequence of numbers that approach ∞. Suppose that $\{\phi_k, k \in \mathbb{N}\}$ solves the Dirichelt boundary value problem of equation 2.5 in $\Sigma_{n_k} \times M$ with boundary data

$$\phi_k(0) = \varphi_0, \qquad \text{and} \qquad \phi_k(n_k) = \rho(n_k). \tag{3.7}$$

Lemma 3.16. *For any smooth geodesic ray $\rho(t)(t \in [0, \infty))$ and for any initial metric $\varphi_0 \in \mathcal{H}$, there exists a uniform constant C such that for any $T \in (0, \infty)$,*

there exists a unique $C^{1,1}$ geodesic $\phi_T(t)(t \in [0,T])$ which connects φ_0 to $\rho(T)$ such that

$$-C \leq \phi_T(t,x) - \rho(t,x) \leq C. \tag{3.8}$$

To obtain uniform $C^{1,1}$ bound in some fashion, we need to choose some appropriate background Kähler metric first. Let h be the ambient metric with bounded ambient geometry. Suppose that this initial geodesic ray $\rho(t)(t \in [0,\infty))$ is tamed by h. Suppose that its Kähler form $\tilde{\omega}$ is given by

$$\pi_2^*\omega_0 + \sum_{i,j=1}^{n} \frac{\partial^2 \bar{\rho}}{\partial w^i \partial w^{\bar{j}}} + 2Re\left(\sum_{i=1}^{n} \frac{\partial^2 \bar{\rho}}{\partial w^i \partial \bar{z}} d\,w^i d\,w^{\bar{j}}\right) + \frac{\partial^2 \bar{\rho}}{\partial z \partial \bar{z}} d\,z\; d\,\bar{z}. \tag{3.9}$$

Here $z = t + \sqrt{-1}\theta$. In other words

$$\tilde{\omega} = \pi_2^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\bar{\rho}.$$

The Dirichelet boundary value problem eq. 3.5 and 3.6 can be re-written as a Drichelet problem on $\Sigma_T \times M$ such that

$$\det\left(h_{\alpha\bar{\beta}} + \frac{\partial^2(\phi - \bar{\rho})}{\partial w^\alpha \partial w^{\bar{\beta}}}\right)_{(n+1)\times(n+1)} = 0, \tag{3.10}$$

with boundary condition

$$\phi\,|_{\{0\}\times S^1\times M} = \varphi_0, \;\; \text{and} \;\; \phi\,|_{\{T\}\times S^1\times M} = \rho(T). \tag{3.11}$$

Set

$$\tilde{\psi}_T(t,x) = \phi_T(t,x) - \bar{\rho}(t,x). \tag{3.12}$$

For a sequence of points $t_i \to \infty$ we have

$$\tilde{\psi}_{t_i}(t_i,x) = \phi_{t_i}(t_i,x) - \bar{\rho}(t_i,x) = \rho_{t_i}(t_i,x) - \bar{\rho}(t_i,x).$$

For simplicity, we drop the dependency on T. Thus, the modified potential $\tilde{\psi}_T(t,x)$ has uniform C^0 bound.

As in [9], we want to use the method of continuity. So we set up the problem as

$$\det\left(h_{\alpha\bar{\beta}} + \frac{\partial^2 \tilde{\psi}}{\partial w^\alpha \partial w^{\bar{\beta}}}\right)_{(n+1)\times(n+1)} = \epsilon \det\left(g_{i\bar{j}}\right)_{n\times n}, \tag{3.13}$$

with boundary condition

$$\tilde{\psi}\,|_{\{0\}\times S^1\times M} = \varphi_0 - \rho(0), \;\; \text{and} \;\; \tilde{\psi}\,|_{\{T\}\times S^1\times M} = \rho(T) - \bar{\rho}(T). \tag{3.14}$$

For any T fixed, this Drichelet boundary value has a unique $C^{1,1}$ solution as in [9]. The challenge at hand is how to obtain a $C^{1,1}$ estimate when T runs over an increasing sequence of times $\{n_k \in \mathbb{N}\}$ and $\epsilon \to 0$.

3.2.2 The $C^{1,1}$ estimates for the HCMA equation in unbounded domains

In this subsection, we want to solve equation 3.13 for any large $T > 0$. We follow Yau's estimate in [33] and we want to set up some notations first. Put $\omega_{\tilde{\rho}(t)} = \sqrt{-1}h_{\alpha\bar{\beta}}d\,w^\alpha \otimes w^{\bar{\beta}}$ and $\omega_{\varphi(t)} = \sqrt{-1}g'_{\alpha\bar{\beta}}d\,w^\alpha \otimes d\,w^{\bar{\beta}}$ where

$$g'_{\alpha\bar{\beta}} = h_{\alpha\bar{\beta}} + \frac{\partial^2\left(\varphi(t) - \bar{\rho}(t)\right)}{\partial w^\alpha \partial w^{\bar{\beta}}}.$$

Then

$$\triangle' = \sum_{\alpha,\beta=1}^{n} {g'}^{\alpha\bar{\beta}} \frac{\partial^2}{\partial w^\alpha \partial w^{\bar{\beta}}}, \qquad \tilde{\triangle} = \sum_{\alpha,\beta=1}^{n} h^{\alpha\bar{\beta}} \frac{\partial^2}{\partial w^\alpha \partial w^{\bar{\beta}}}.$$

Before stating the crucial Lemma of this subsection, we need to explain a little bit the relationship between geodesic ray and its bounded ambient geometry. By definition of the initial geodesci ray tamed by an ambient metric h with bounded ambient geometry, there exists a sequence of time $t_i \to \infty$ such that

1. $|n + 1 + \triangle_h(\rho - \bar{\rho})| \leq C$;
2. $|\frac{\partial(\rho-\bar{\rho})}{\partial t}|_h \leq C$;
3. The vector $|\frac{\partial}{\partial t}|_h$ has uniform upper bound.

Note that under the first two conditions, $\rho - \bar{\rho}$ is not necessary bounded. However, it is sufficient to show that the oscillation of $|\rho - \bar{\rho}|$ is controlled by the distance (by ambient metric h). We first prove the following lemma:

Lemma 3.17. *[9] There exists a constant C which depends only on the ambient metric h (independent of T) such that*

$$e^{-\lambda(\rho-\bar{\rho})}(n + 1 + \tilde{\triangle}\tilde{\psi}(t)) \leq \max_{t=0,t=T} e^{-\lambda(\rho-\bar{\rho})}(n + 1 + \tilde{\triangle}\tilde{\psi}(t)).$$

Proof. We want to use the maximum principle in this proof. Let us first calculate $\triangle'\left(n + 1 + \tilde{\triangle}(\varphi - \bar{\rho})\right)$.

Let us choose a coordinate so that at a fixed point both $\omega_{\tilde{\rho}(t)} = \sqrt{-1}h_{\alpha\bar{\beta}}d\,w^\alpha \otimes d\,w^{\bar{\beta}}$ and the complex Hessian of $\varphi(t) - \rho(t)$ are in diagonal forms. In particular, we assume that $h_{i\bar{j}} = \delta_{i\bar{j}}$ and $(\varphi(t) - \rho(t))_{i\bar{j}} = \delta_{i\bar{j}}\,(\varphi(t) - \rho(t))_{i\bar{i}}$. Thus

$${g'}^{i\bar{s}} = \frac{\delta_{i\bar{s}}}{1 + (\varphi(t) - \rho(t))_{i\bar{i}}}.$$

For convenience, put

$$F = \ln\epsilon + \log\det\left(h_{i\bar{j}}\right).$$

Then our equation reduces to

$$\log\det\left(h_{i\bar{j}} + \frac{\partial^2\left(\varphi - \bar{\rho}\right)}{\partial w_i \partial w_{\bar{j}}}\right) = F + \log\det(h_{i\bar{j}}).$$

For convenience, set

$$\tilde{\psi}(t) = \varphi(t) - \bar{\rho}(t)$$

in this proof. Note that $|\tilde{\psi}(t)|$ is uniformly bounded. We first follow the standard calculation of C^2 estimates in [33]. Differentiate both sides with respect to $\frac{\partial}{\partial w_k}$

$$(g')^{i\bar{j}}\left(\frac{\partial h_{i\bar{j}}}{\partial w_k} + \frac{\partial^3 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k}\right) - h^{i\bar{j}}\frac{\partial h_{i\bar{j}}}{\partial w_k} = \frac{\partial F}{\partial w_k},$$

and differentiating again with respect to $\frac{\partial}{\partial \bar{w}_l}$ yields

$$\begin{aligned}
&(g')^{i\bar{j}}\left(\frac{\partial^2 h_{i\bar{j}}}{\partial w_k \partial \bar{w}_l} + \frac{\partial^4 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k \partial \bar{w}_l}\right) + h^{t\bar{j}} h^{i\bar{s}} \frac{\partial h_{t\bar{s}}}{\partial \bar{w}_l}\frac{\partial h_{i\bar{j}}}{\partial w_k} - h^{i\bar{j}}\frac{\partial^2 h_{i\bar{j}}}{\partial w_k \partial \bar{w}_l} \\
&-(g')^{t\bar{j}}(g')^{i\bar{s}}\left(\frac{\partial h_{t\bar{s}}}{\partial \bar{w}_l} + \frac{\partial^3 \tilde{\psi}(t)}{\partial w_t \partial \bar{w}_s \partial \bar{w}_l}\right)\left(\frac{\partial h_{i\bar{j}}}{\partial w_k} + \frac{\partial^3 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k}\right) = \frac{\partial^2 F}{\partial w_k \partial \bar{w}_l}.
\end{aligned}$$

Assume that we have normal coordinates at the given point, i.e., $h_{i\bar{j}} = \delta_{ij}$ and the first order derivatives of g vanish. Now taking the trace of both sides results in

$$\begin{aligned}
\widetilde{\Delta} F = {}& h^{k\bar{l}}(g')^{i\bar{j}}\left(\frac{\partial^2 h_{i\bar{j}}}{\partial w_k \partial \bar{w}_l} + \frac{\partial^4 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k \partial \bar{w}_l}\right) \\
&- h^{k\bar{l}}(g')^{t\bar{j}}(g')^{i\bar{s}}\frac{\partial^3 \tilde{\psi}(t)}{\partial w_t \partial \bar{w}_s \partial \bar{w}_l}\frac{\partial^3 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k} - h^{k\bar{l}} h^{i\bar{j}}\frac{\partial^2 h_{i\bar{j}}}{\partial w_k \partial \bar{w}_l}.
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
\Delta'(\widetilde{\Delta}\tilde{\psi}(t)) &= (g')^{k\bar{l}}\frac{\partial^2}{\partial w_k \partial \bar{w}_l}\left(h^{i\bar{j}}\frac{\partial^2 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j}\right) \\
&= (g')^{k\bar{l}} h^{i\bar{j}}\frac{\partial^4 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k \partial \bar{w}_l} + (g')^{k\bar{l}}\frac{\partial^2 h^{i\bar{j}}}{\partial w_k \partial \bar{w}_l}\frac{\partial^2 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j},
\end{aligned}$$

and we will substitute $\frac{\partial^4 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k \partial \bar{w}_l}$ in $\Delta'(\widetilde{\Delta}\tilde{\psi}(t))$ so that the above reads

$$\begin{aligned}
\Delta'(\widetilde{\Delta}\tilde{\psi}(t)) = {}& -h^{k\bar{l}}(g')^{i\bar{j}}\frac{\partial^2 h_{i\bar{j}}}{\partial w_k \partial \bar{w}_l} + h^{k\bar{l}}(g')^{t\bar{j}}(g')^{i\bar{s}}\frac{\partial^3 \tilde{\psi}(t)}{\partial w_t \partial \bar{w}_s \partial \bar{w}_l}\frac{\partial^3 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k} \\
&+ h^{k\bar{l}} h^{i\bar{j}}\frac{\partial^2 h_{i\bar{j}}}{\partial w_k \partial \bar{w}_l} + \widetilde{\Delta} F + (g')^{k\bar{l}}\frac{\partial^2 h^{i\bar{j}}}{\partial w_k \partial \bar{w}_l}\frac{\partial^2 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j},
\end{aligned}$$

which we can rewrite after substituting $\frac{\partial^2 h_{i\bar{j}}}{\partial w_k \partial \bar{w}_l} = -R_{i\bar{j}k\bar{l}}$ and $\frac{\partial^2 h^{i\bar{j}}}{\partial w_k \partial \bar{w}_l} = R_{j\bar{i}k\bar{l}}$ as

$$\begin{aligned}
\Delta'(\widetilde{\Delta}\tilde{\psi}(t)) \quad = \quad & \widetilde{\Delta} F + h^{k\bar{l}}(g')^{t\bar{j}}(g')^{i\bar{s}}\tilde{\psi}(t)_{t\bar{s}l}\tilde{\psi}(t)_{i\bar{j}k} \\
& +(g')^{i\bar{j}} h^{k\bar{l}} R_{i\bar{j}k\bar{l}} - h^{i\bar{j}} h^{k\bar{l}} R_{i\bar{j}k\bar{l}} + (g')^{k\bar{l}} R_{j\bar{i}k\bar{l}}\tilde{\psi}(t)_{i\bar{j}}.
\end{aligned}$$

Restrict to the coordinates we chose in the beginning so that both g and $\tilde{\psi}(t)$ are in diagonal form. The above transforms to

$$\begin{aligned}\Delta'(\widetilde{\Delta}\tilde{\psi}(t)) &= \tfrac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\tfrac{1}{1+\tilde{\psi}(t)_{j\bar{j}}}\tilde{\psi}(t)_{i\bar{j}k}\tilde{\psi}(t)_{\bar{i}j\bar{k}}+\widetilde{\Delta}F \\ &\qquad +R_{i\bar{i}k\bar{k}}(-1+\tfrac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}+\tfrac{\tilde{\psi}(t)_{i\bar{i}}}{1+\tilde{\psi}(t)_{k\bar{k}}}).\end{aligned}$$

Set now $C=\inf_{i\neq k}R_{i\bar{i}k\bar{k}}$ and observe that

$$\begin{aligned}R_{i\bar{i}k\bar{k}}(-1+\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}+\frac{\tilde{\psi}(t)_{i\bar{i}}}{1+\tilde{\psi}(t)_{k\bar{k}}}) &= \frac{1}{2}R_{i\bar{i}k\bar{k}}\frac{(\tilde{\psi}(t)_{k\bar{k}}-\tilde{\psi}(t)_{i\bar{i}})^2}{(1+\tilde{\psi}(t)_{i\bar{i}})(1+\psi(t)_{k\bar{k}})} \\ &\geq \frac{C}{2}\frac{(1+\tilde{\psi}(t)_{k\bar{k}}-1-\psi(t)_{i\bar{i}})^2}{(1+\tilde{\psi}(t)_{i\bar{i}})(1+\tilde{\psi}(t)_{k\bar{k}})} \\ &= C\biggl(\frac{1+\tilde{\psi}(t)_{i\bar{i}}}{1+\tilde{\psi}(t)_{k\bar{k}}}-1\biggr),\end{aligned}$$

which yields

$$\begin{aligned}\Delta'(\widetilde{\Delta}\tilde{\psi}(t)) &\geq \tfrac{1}{(1+\tilde{\psi}(t)_{i\bar{i}})(1+\tilde{\psi}(t)_{j\bar{j}})}\tilde{\psi}(t)_{i\bar{j}k}\tilde{\psi}(t)_{\bar{i}j\bar{k}}+\widetilde{\Delta}F \\ &\qquad +C\biggl((n+1+\widetilde{\Delta}\tilde{\psi}(t))\textstyle\sum_i\tfrac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}-1\biggr).\end{aligned}$$

We need to apply one more trick to obtain the requested estimates. Namely,

$$\begin{aligned}&\Delta'(e^{-l\tilde{\psi}(t)}(n+1+\widetilde{\Delta}\tilde{\psi}(t))) \\ =\;& e^{-l\tilde{\psi}(t)}\Delta'(\widetilde{\Delta}\psi(t))+2\nabla' e^{-l\tilde{\psi}(t)}\nabla'(n+\widetilde{\Delta}\psi(t)) \\ &\quad +\Delta'(e^{-l\tilde{\psi}(t)})(n+1+\widetilde{\Delta}\tilde{\psi}(t)) \\ =\;& e^{-l\tilde{\psi}(t)}\Delta'(\widetilde{\Delta}\tilde{\psi}(t))-le^{-l\tilde{\psi}(t)}(g')^{i\bar{i}}\tilde{\psi}(t)_i(\widetilde{\Delta}\tilde{\psi}(t))_{\bar{i}} \\ &\quad -le^{-l\tilde{\psi}(t)}(g')^{i\bar{i}}\tilde{\psi}(t)_{\bar{i}}(\widetilde{\Delta}\tilde{\psi}(t))_i \\ &\quad -le^{-l\tilde{\psi}(t)}\Delta'\tilde{\psi}(t)(n+1+\widetilde{\Delta}\tilde{\psi}(t)) \\ &\quad +l^2e^{-l\tilde{\psi}(t)}(g')^{i\bar{i}}\tilde{\psi}(t)_i\tilde{\psi}(t)_{\bar{i}}(n+1+\widetilde{\Delta}\tilde{\psi}(t)) \\ \geq\;& e^{-l\tilde{\psi}(t)}\Delta'(\widetilde{\Delta}\tilde{\psi}(t)) \\ &-e^{-l\tilde{\psi}(t)}(g')^{i\bar{i}}(n+\widetilde{\Delta}\tilde{\psi}(t))^{-1}(\widetilde{\Delta}\tilde{\psi}(t))_i(\widetilde{\Delta}\tilde{\psi}(t))_{\bar{i}} \\ &\quad -le^{-l\tilde{\psi}(t)}\Delta'\tilde{\psi}(t)(n+1+\widetilde{\Delta}\tilde{\psi}(t)),\end{aligned}$$

which follows from the Schwarz Lemma applied to the middle two terms. We will write out one term here; the other goes in an analogous way.

$$\begin{aligned}&(le^{-\frac{l}{2}\tilde{\psi}(t)}\tilde{\psi}(t)_i(n+\widetilde{\Delta}\tilde{\psi}(t))^{\frac{1}{2}})(e^{-\frac{l}{2}\tilde{\psi}(t)}(\widetilde{\Delta}\tilde{\psi}(t))_{\bar{i}}(n+1+\widetilde{\Delta}\tilde{\psi}(t))^{-\frac{1}{2}}) \\ \leq&\frac{1}{2}(l^2e^{-l\tilde{\psi}(t)}\tilde{\psi}(t)_i\tilde{\psi}(t)_{\bar{i}}(n+1+\widetilde{\Delta}\tilde{\psi}(t)) \\ &+e^{-l\tilde{\psi}(t)}(\widetilde{\Delta}\tilde{\psi}(t))_{\bar{i}}(\widetilde{\Delta}\tilde{\psi}(t))_i(n+\widetilde{\Delta}\tilde{\psi}(t))^{-1}).\end{aligned}$$

Consider now the following

$$
\begin{aligned}
&-(n+1+\widetilde{\Delta}\tilde{\psi}(t))^{-1}\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}(\widetilde{\Delta}\tilde{\psi}(t))_i(\widetilde{\Delta}\tilde{\psi}(t))_{\bar{i}}+\Delta'\widetilde{\Delta}\tilde{\psi}(t)\geq\\
&-(n+1+\widetilde{\Delta}\tilde{\psi}(t))^{-1}\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}|\tilde{\psi}(t)_{k\bar{k}i}|^2+\widetilde{\Delta}F\\
&+\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\frac{1}{1+\tilde{\psi}(t)_{k\bar{k}}}\tilde{\psi}(t)_{k\bar{i}\bar{j}}\tilde{\psi}(t)_{i\bar{k}j}+C(n+1+\widetilde{\Delta}\tilde{\psi}(t))\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}.
\end{aligned}
$$

On the other hand, using the Schwarz inequality, we have

$$
\begin{aligned}
&(n+\widetilde{\Delta}\tilde{\psi}(t))^{-1}\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}|\tilde{\psi}(t)_{k\bar{k}i}|^2\\
&\quad=(n+1+\widetilde{\Delta}\tilde{\psi}(t))^{-1}\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\left|\frac{\tilde{\psi}(t)_{k\bar{k}i}}{(1+\tilde{\psi}(t)_{k\bar{k}})^{\frac{1}{2}}}(1+\tilde{\psi}(t)_{k\bar{k}})^{\frac{1}{2}}\right|^2\\
&\quad\leq(n+1+\widetilde{\Delta}\psi(t))^{-1}\left(\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\frac{1}{1+\tilde{\psi}(t)_{k\bar{k}}}\tilde{\psi}(t)_{k\bar{k}i}\tilde{\psi}(t)_{\bar{k}k\bar{i}}\right)\left(1+\tilde{\psi}(t)_{l\bar{l}}\right)\\
&\quad=\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\frac{1}{1+\tilde{\psi}(t)_{k\bar{k}}}\tilde{\psi}(t)_{k\bar{k}i}\tilde{\psi}(t)_{\bar{k}k\bar{i}}\\
&\quad=\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\frac{1}{1+\tilde{\psi}(t)_{k\bar{k}}}\psi(t)_{i\bar{k}k}\tilde{\psi}(t)_{k\bar{i}\bar{k}}\\
&\quad\leq\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\frac{1}{1+\tilde{\psi}(t)_{k\bar{k}}}\tilde{\psi}(t)_{i\bar{k}j}\tilde{\psi}(t)_{k\bar{i}\bar{j}},
\end{aligned}
$$

so that we get

$$
\begin{aligned}
&-(n+\widetilde{\Delta}\tilde{\psi}(t))^{-1}\tfrac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}(\widetilde{\Delta}\tilde{\psi}(t))_i(\widetilde{\Delta}\tilde{\psi}(t))_{\bar{i}}+\Delta'\widetilde{\Delta}\tilde{\psi}(t)\\
&\geq\widetilde{\Delta}F+C(n+1+\widetilde{\Delta}\tilde{\psi}(t))\tfrac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}.
\end{aligned}
$$

Putting all these together, we obtain

$$
\begin{aligned}
&\triangle'\left(e^{-\lambda\tilde{\psi}(t)}(n+\tilde{\triangle}\tilde{\psi}(t))\right)\\
&\geq e^{-\lambda\tilde{\psi}(t)}\left(\tilde{\triangle}F+C(n+1+\tilde{\triangle}\tilde{\psi}(t))\sum_{i=1}^{n}\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\right)\\
&\qquad-\lambda\, e^{-\lambda\tilde{\psi}(t)}\triangle'\tilde{\psi}(t)\,(n+1+\tilde{\triangle}\tilde{\psi}).
\end{aligned}
\tag{3.15}
$$

Consider

$$
\tilde{\triangle}F \quad=\quad h^{\alpha\bar{\beta}}\frac{\partial^2\log\det(h_{i\bar{j}})}{\partial w^\alpha\partial w^{\bar{\beta}}}=-R(\rho(t)).
$$

Plugging this into the inequality (3.15), we obtain

$$
\begin{aligned}
&\triangle' \left(e^{-\lambda\tilde{\psi}(t)} (n + \tilde{\triangle}\psi(t)) \right) \\
&\geq e^{-\lambda\tilde{\psi}(t)} \left(C(n+1+\tilde{\triangle}\psi(t)) \sum_{i=1}^{n} \frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}} \right) \\
&\qquad -\lambda\, e^{-\lambda\tilde{\psi}(t)} \triangle'\tilde{\psi}(t)\ (n+1+\tilde{\triangle}\tilde{\psi}(t)) - R(\rho(t))\, e^{-\lambda\tilde{\psi}(t)}.
\end{aligned}
$$

Now

$$
\begin{aligned}
\triangle'\tilde{\psi}(t) &= \triangle'\tilde{\psi}(t) = tr_{g'}(\tilde{\omega} + i\partial\bar{\partial}\tilde{\psi} - (\tilde{\omega}) \\
&= n+1 - tr_{g'} h.
\end{aligned}
$$

Plugging this into the above inequality, we obtain

$$
\begin{aligned}
&\triangle' \left(e^{-\lambda\tilde{\psi}(t)} (n + \tilde{\triangle}\tilde{\psi}(t)) \right) \\
&\geq e^{-\lambda\tilde{\psi}(t)} \left((C+\lambda\delta)(n+1+\tilde{\triangle}\tilde{\psi}(t)) \sum_{i=1}^{n+1} \frac{1}{1+\psi(t)_{i\bar{i}}} \right) \\
&\qquad -|\lambda| c_3\, e^{-\lambda\tilde{\psi}(t)}\ (n+1+\tilde{\triangle}\tilde{\psi}(t)) - R(\rho(t))\, e^{-\lambda\tilde{\psi}(t)}.
\end{aligned}
$$

Let $\lambda\delta = -C + 1$, we then have

$$
\begin{aligned}
&\triangle' \left(e^{-\lambda\tilde{\psi}(t)} (n + \tilde{\triangle}\tilde{\psi}(t)) \right) \\
&\geq e^{-\lambda\tilde{\psi}(t)} \left((n+\tilde{\triangle}\tilde{\psi}(t)) \sum_{i=1}^{n+1} \frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}} \right) \\
&\qquad -c_5\, e^{-\lambda\tilde{\psi}(t)}\ (n+1+\tilde{\triangle}\tilde{\psi}(t)) - c_2\, e^{-\lambda\tilde{\psi}}.
\end{aligned}
$$

Here c_5 is a uniform constant.

Claim: the maximum value of $e^{-\lambda\tilde{\psi}(t)}(n+1+\tilde{\triangle}\tilde{\psi}(t))$ must occur in $\partial\Sigma_T \times M$.

Otherwise, if the maximum occur in the interior, we have

$$
e^{-\lambda\tilde{\psi}(t)} \left((n+1+\tilde{\triangle}\tilde{\psi}(t)) \sum_{i=1}^{n+1} \frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}} \right) - c_5\, e^{-\lambda\tilde{\psi}(t)}\ (n+1+\tilde{\triangle}\tilde{\psi}(t)) - c_2\, e^{-\lambda\tilde{\psi}} \leq 0.
$$

However,

$$
\sum_{i=1}^{n+1} \frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}} \to \infty
$$

as $\epsilon \to 0$. This leads to a contradiction when T is finite since

$$
\tilde{\psi} = \rho - \bar{\rho} + \varphi_T - \rho
$$

is uniformly bounded in $\Sigma_T \times M$. Thus,

$$
e^{-\lambda\tilde{\psi}(t)}(n+1+\tilde{\triangle}\tilde{\psi}(t)) \leq \max_{t=0,t=T}\ e^{-\lambda\tilde{\psi}(t)}(n+1+\tilde{\triangle}\tilde{\psi}(t)).
$$

In other words,

$$e^{-\lambda(\rho-\bar{\rho})}(n+1+\tilde{\triangle}\tilde{\psi}(t)) \leq \max_{t=0,t=T} e^{-\lambda(\rho-\bar{\rho})}(n+1+\tilde{\triangle}\tilde{\psi}(t)).$$

Note that

$$|\tilde{\triangle}(\rho-\bar{\rho})| + |\frac{\partial(\rho-\bar{\rho})}{\partial t}| \leq C.$$

This implies Lemma 3.17 (cc. [9]) $\square$

As in [9], we have

Theorem 3.18. *[9] If ψ is a solution of equation (3.13) at $0<\epsilon<1$, then there exists a constant C which depends only on $(\Sigma_T \times M, h)$ such that if $e^{-\lambda(\rho-\bar{\rho})}(n+1+\tilde{\triangle}\tilde{\psi}(t))$ attains the maximal value at $t=T$, then for any $\{t\}\times S^1\times M$ which has $h-$ distance to $\{T\}\times S^1\times M$ less than 1, we have*

$$\max_{\{t\}\times S^1\times M}(n+1+\tilde{\triangle}\tilde{\psi}) \leq C \max_{[T-\mu,T]\times S^1\times M}(|\nabla\tilde{\psi}|_h^2+1), \qquad (3.16)$$

hold for any $\mu>0$ where the h distance from $\{T-\mu\}\times S^1\times M$ to $\{T\}\times M$ is small ($\ll 1$). On the other hand, if $e^{-\lambda(\rho-\bar{\rho})}(n+1+\tilde{\triangle}\tilde{\psi}(t))$ attains the maximal value at $t=0$, then

$$\max_{[0,1]\times S^1\times M}(n+1+\tilde{\triangle}\tilde{\psi}) \leq C \max_{[0,\mu]\times S^1\times M}(|\nabla\tilde{\psi}|_h^2+1). \qquad (3.17)$$

For simplicity, denote the h distance between two hypersurfaces $\{t-1\}\times S^1\times M$ and $\{t_2\}\times S^1\times M$ as $d_h(t_1,t_2)$. Following a blowing up argument in [9], we can prove that there is a uniform $C^{1,1}$ estimate for $t\in[0,T]$ or $t\in[0,1]$, depending on where

$$e^{-\lambda\tilde{\psi}(t)}(n+1+\tilde{\triangle}\tilde{\psi}(t))$$

realizes its maximum. For simplicity, let us assume that

$$\eta^{-\lambda\tilde{\psi}(t)}(n+1+\tilde{\triangle}\tilde{\psi}(t))$$

obtain maximum at $\{T\}\times S^1\times M$. Thus, we have

$$\max_{\{t\}\times S^1\times M}(n+1+\tilde{\triangle}\tilde{\psi}) \leq C \max_{[T-\mu,T]\times S^1\times M}(|\nabla\tilde{\psi}|_h^2+1), \qquad \forall \mu>0$$

for any t where the $d_h(t,T)\leq 1$ and $d_h(T-\mu,T)\ll 1$. Unlike in [9], here we need to blowup at the maximam point of

$$|\nabla\tilde{\psi}|_h \cdot (\frac{1}{2}-d_h(t,T)), \qquad \forall t\in[\frac{T}{2},T].$$

Note that we don't assume that $\tilde{\psi}$ has a uniform C^0 bound. To bypass this difficulty, we note that

$$\tilde{\psi} = \varphi-\rho+\rho-\bar{\rho}.$$

By assumption, the first and second derivatives of the two functions $\tilde{\psi}$ and $\varphi - \rho$ are equivalent. Therefore, we really blowup at the maximum of

$$|\nabla(\varphi - \rho)|_h \cdot t(\frac{1}{2} - d_h(t,T)), \qquad \forall t \in [\frac{T}{2}, T].$$

As in [9], we can prove that $|\nabla(\varphi - \rho)|_h$ is uniformly bounded for $t \in [T - \frac{1}{2}, T]$. Consequently, this implies uniform control on $|\nabla\tilde{\psi}|_h$. The crucial observation here is that the distance function $d_h(t,T)$ is a positive function which vanishes in the boundary and the hessian of $d_h(t,T)$ with respect to the metric h is positive and bounded.

Using Theorem 3.18, we have

$$n + 1 + \tilde{\triangle}\tilde{\psi} \leq C$$

where C is a constant independent of T. Following estimates of Lemma 3.17, we obtain a uniform growth control on geodesic ray. Theorem 3.8 then follows.

4 On the lower bound of the Calabi energy

In this section, we will give an lower bound estimate for the modified Calabi energy in absence of a cscK metrc or an extremal Kähler metric. Note that for algebraic manifold, the corresponding theorem is given in [17].

4.1 The classic theory of Futaki-Mabuchi and A. Hwang

Let $K = K(J)$ be a maximal compact subgroup of the automorphism group of the Kähler manifold and let $\mathcal{K} = \mathcal{K}(J)$ be its Lie algebra of gradient holomorphic vector fields in M. According to E. Calabi, if there is a cscK metric of CextrK metric, the cscK metric or CextrK metric must be symmetric with respect to one of these maximal compact subgroup (up to holomorphic conjugation). Therefore, it makes perfect sense to consider a restricted class $\mathcal{H}_K \subset \mathcal{H}$ where all Kähler metrics are invariant under K. For simplicity, suppose ω is invariant with respect to action of K. Recalled that the Lichenowicz operator is defined as:

$$\mathcal{L}_g(f) = f_{,\alpha\beta} d\, z^\alpha \otimes d\, z^\beta.$$

where the right hand side is (2,0) component of the Hessian form of f with respect to the Kähler metric g. For any metric $g \in \mathcal{H}_K$, define $Ker(\mathcal{L}_g)$ as the real part of the Kernel subspace[8] of the operator $\mathcal{L}_g$ in $C^\infty(M)$. It is easy to see the correspondence between $Ker(\mathcal{L}_g)$ and $\mathcal{K}$ in the following formula

$$X = g^{\alpha\bar{\beta}} \frac{\partial}{\partial w^\alpha} \frac{\partial \theta_X}{\partial w^{\bar{\beta}}}$$

[8]Usually, the Kernal space can not be split as real part and imaginary part. However, in the case when the metric is invariant under $K(J)$, its Lie algebra $\mathcal{K}(J)$ always corresponds to this real part of the Kernel space.

where

$$X \in \mathcal{K}, \theta_X \in Ker(\mathcal{L}_g) \qquad \text{and} \quad \int_M \theta_X \; \omega_g^n = 0.$$

It is easy to see that such a correspondence is 1-1 as long as $g \in \mathcal{K}$. Futaki-Mabuchi define a bilinear form in $\mathcal{K}$ by

$$(X, Y) = \int_M \theta_X \theta_Y \omega_g^n.$$

Here θ_Y is the holomorphic potential of Y. Futaki-Mabuchi proved that such a bilinear form is positive definite and well defined for $\mathcal{H}_K$. From the definition of θ_X, it is easy to see that if $g \in \mathcal{H}_K$, then θ_X is real since

$$L_{Im(X)} \omega_g = 0, \qquad \forall\; X \in \mathcal{K}, \text{and } \; g \in \mathcal{H}_K.$$

Thus, the Futaki-Mabuchi bilinear form is positive definite. To show it is well defined, we need to show that it is invariant when metrics varies inside $\mathcal{H}_K$. Let $\omega_{g(t)} = \omega_g + t\sqrt{-1}\partial\bar{\partial}\varphi \in \mathcal{H}_K$. Let

$$X = g(t)^{\alpha\bar{\beta}} \frac{\partial}{\partial w^\alpha} \frac{\partial \theta_X(t)}{\partial w^{\bar{\beta}}}, \;\; \text{and}$$

$$Y = g(t)^{\alpha\bar{\beta}} \frac{\partial}{\partial w^\alpha} \frac{\partial \theta_Y(t)}{\partial w^{\bar{\beta}}}.$$

Then,

$$\theta_X(t) = \theta_X + tX(\varphi), \qquad \theta_Y(t) = \theta_Y + tY(\varphi),$$

where

$$L_{Im(X)}\varphi = L_{Im(Y)}\varphi = 0.$$

Set

$$(X, Y)_t = \int_M \theta_X(t)\theta_Y(t)\omega_{g(t)}^n.$$

It is straightforward to compute

$$\begin{aligned}
\tfrac{d}{dt}(X,Y)_t &= \int_M \left(\theta_Y(t)\frac{d}{dt}\theta_X(t) + \theta_X(t)\frac{d}{dt}\theta_Y(t) + \theta_X(t)\theta_Y(t)\triangle_{g(t)}\varphi\right)\omega_{g(t)}^n \\
&= \int_M (\theta_Y(t)X(\varphi) + \theta_X(t)Y(\varphi) \\
&\qquad -g(t)^{\alpha\bar{\beta}}\tfrac{\partial\theta_X(t)}{\partial w^{\beta}}Y(\varphi)\tfrac{\partial\varphi}{\partial w^\alpha} - g(t)^{\alpha\bar{\beta}}\tfrac{\partial\theta_Y(t)}{\partial w^{\beta}}X(\varphi)\tfrac{\partial\varphi}{\partial w^\alpha}\Big)\omega_{g(t)}^n \\
&= \int_M (\theta_Y(t)X(\varphi) + \theta_X(t)Y(\varphi) - \theta_Y(t)X(\varphi) - \theta_X(t)Y(\varphi))\,\omega_{g(t)}^n \\
&= 0.
\end{aligned}$$

Thus, the Futaki-Mabuchi bilinear form is well defined. Now, Futaki character defines a linear map from $\mathcal{K}$ to R, by Rezzi representation formula, there is a unique vector field $\mathcal{X}_c \in \mathcal{K}$ such that

$$\mathcal{F}_X([\omega]) = (X, \mathcal{X}_c), \qquad \forall X \in \mathcal{K}.$$

Since both $\mathcal{F}_X$ and the Futaki-Mabuchi form is independent of the metric, so is $\mathcal{X}_c$ *a priori*. When there is an extremal Kähler metric, then $\mathcal{X}_c$ coincides with the complex gradient vector field of the scalar curvature function.

Theorem 4.1. *(Hwang) The following inequality holds*

$$\inf_{g\in\mathcal{H}_K} Ca(\omega_g) \geq \mathcal{F}_{\mathcal{X}_c}([\omega]),$$

where the equality holds if there is an extremal Kähler metric in $[\omega]$.

Proof. Suppose $g \in \mathcal{K}$ and using the L^2 norm with respect to ω_g^n to decompose $R(g) - \bar{R}$ as

$$R(g) - \bar{R} = -\rho - \rho^{\perp} = -\triangle_g F, \qquad \text{where } \ \rho \in Ker(\mathcal{L}_g),$$

it is easy to see that

$$\mathcal{X}_c = g^{\alpha\bar{\beta}} \frac{\partial}{\partial w^{\alpha}} \frac{\partial \rho}{\partial w^{\bar{\beta}}}.$$

Thus,

$$\begin{aligned} Ca(\omega_g) &= \int_M (R(g) - \bar{R})^2 \omega_g^n \\ &= \int_M \rho^2 \omega_g^n + \int_M (\rho^{\perp})^2 \omega_g^n \\ &\geq -\int_M \rho(R(g) - \bar{R}) \omega_g^n = \int_M \rho \triangle_g F \omega_g^n = -\int_M \nabla\rho \cdot \nabla F \omega_g^n \\ &= \int_M \mathcal{X}_c(F) \omega_g^n = \mathcal{F}_{\mathcal{X}_c}([\omega]). \end{aligned}$$

□

At the time, Andrew Hwang thought the same proof could be extended to cover the non-invariant case. Unfortunately, the Futaki-Mabuchi form is no longer positive definite and the whole argument collapsed. Much efforts have been made by other mathematicians to bridge the gap, none are succsful. Nonetheless, it is very interesting and also important to generalize the above theorem to a more general settings.

4.2 The first derivatives of the K energy

It is well known that the first derivatives of the K energy functional is monotonically increasing along any smooth geodesic segment or ray. Using the latest result of Chen-Tian [12], we can show that the difference of the first derivatives of the K energy at the two ends of any $C^{1,1}$ geodesic segment always have a preferred sign. This property turns out to be sufficient for our purpose.

For any two Kähler potentials $\phi_0, \phi_1 \in \mathcal{H}$, we want to use the almost smooth solution to approximate the $C^{1,1}$ geodesic between ϕ_0 and ϕ_1. This is an approach first taken in [12]. For any integer l, consider Drichelet problem for the

HCMA equation 2.5 on the rectangle domain $\Sigma_l = [-l, l] \times [0,1]$ with boundary value

$$\phi(s,0) = \phi_0, \phi(s,1) = \phi_1; \qquad \phi(\pm l, t) = (1-t)\phi_0 + (1-t)\phi_1, \qquad (s,t) \in \Sigma_l. \tag{4.1}$$

We may modify this boundary map in the four corners so that the domain is smooth without corner. Denote the almost smooth solution by $\phi^{(l)} : \Sigma_l \to \mathcal{H}$ which corresponds to this boundary map[9]. According to [9], $\phi^{(l)}$ has a uniform $C^{1,1}$ upper bound which is independent of l. Set

$$\mathbf{E}^{(l)}(s,t) = \mathbf{E}(\phi^{(l)}(s,t)), \qquad \forall\ (s,t) \in \Sigma_l. \tag{4.2}$$

and

$$\mathbf{E}(s,t) = \mathbf{E}(\varphi(t)), \qquad \forall\ (s,t) \in S^1 \times [0,1] \times M. \tag{4.3}$$

Before we prove the main theorem, we need a convergence lemma

Lemma 4.2. *For any $m > 0$ fixed, we have $\{\phi^{(l)}(s,t), l \in \mathbf{N}\}$ converges uniformly to φ in the weak $C^{1,1}$ topology. In particular,*

$$\lim_{l\to\infty} \frac{\partial \phi^{(l)}}{\partial s} = 0, \qquad \forall\ \ (s,t) \in \Sigma^{(m)}$$

with respect to any $C^\alpha(0 < \alpha < 1)$ norm in $\Sigma^{(m)} \times M$.

Proof. Note first that ϕ^l has uniform $C^{1,1}$ bound on $\Sigma^{(m)} \times M$. Thus, passing to a subsequence if necessary, we have $\phi^{(l)} \to \bar{\varphi}$ strongly in $C^{1,\alpha}(\alpha \in (0,1))$ or $W^{2,p}(p \in (0,\infty))$ in $\Sigma^{(m)} \times M$ for any m fixed. On the other hand, $\frac{\partial \phi^{(l)}}{\partial s}$ satisfies the following equation:

$$\triangle_z \frac{\partial \phi^{(l)}}{\partial s} = 0$$

where $\triangle_z$ represents the Lapalacian operator along each holomorphic leaf. In other words, $\frac{\partial \phi^{(l)}}{\partial s}$ is a harmonic function which vanishes in main part of the boundary:

$$\frac{\partial \phi^{(l)}}{\partial s}(s,1) = \frac{\partial \phi^{(l)}}{\partial s}(s,0) = 0, \qquad \forall\ s \in [-l+1, l-1].$$

We claim that $\frac{\partial \bar{\varphi}}{\partial s} = 0$. Note that $\frac{\partial \phi^{(l)}}{\partial s} \to \frac{\partial \bar{\varphi}}{\partial s}$ in any $C^\alpha(\Sigma^{(m)} \times M)$ norm. Picking any point $(z_0, x_0) \in \Sigma^{(m)} \times M$, we have

$$\lim_{l\to\infty} \frac{\partial \phi^{(l)}}{\partial s} = \frac{\partial \bar{\varphi}^{(l)}}{\partial s}(z_0, x_0).$$

Using this fact, we can consider the pull back function of $\frac{\partial \phi^{(l)}}{\partial s}$ on each long strip. Suppose $\mathcal{S}_l$ is a holomorphic leaf in $\Sigma_l \times M$ which passes through the

[9] We may need to alter the boundary value slightly.

point (z_0, x_0). Let $\jmath_l : \Sigma^{(l)} \to \Sigma^{(l)} \times M$ be the holomorphic map associated with this leaf $\mathcal{S}_l$. Let h_l be the pulled back function of $\frac{\partial \phi^{(l)}}{\partial s}$ under this sequence of holomorphic maps. For simplicity, assume the preimage of (z_0, x_0) is always $(0, \frac{1}{2})$.

Now $\{h_l, l \in \mathbf{N}\}$ is a sequence of bounded harmonic functions defined in $\Sigma^{(l)}$ such that it vanishes completely in $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$. Note that

$$\Sigma^{(1)} \subset \Sigma^{(2)} \subset \cdots \Sigma^{(l)} \subset \cdots \subset (-\infty, \infty) \times [0,1].$$

Since this sequence of functions vanishes in any fixed compact subset of $\partial \Sigma^{(l)}$ and the limiting domain is an infinitely long strip, a careful argument using the maximum principle will imply that h_l converges strongly to 0 in any compact subset of the infinite strip. In particular, we have

$$\lim_{l \to \infty} h_l(z_0, x_0) = 0$$

or

$$\frac{\partial \bar{\varphi}}{\partial s}(z_0, x_0) = \lim_{l \to \infty} \frac{\partial \phi^{(l)}}{\partial s}(z_0, x_0) = \lim_{l \to \infty} h_l(0, \frac{1}{2}) = 0,$$

since (z_0, x_0) is an arbitrary point in $\Sigma^{(m)} \times M$ for any m fixed. We then prove that $\frac{\partial \phi^{(l)}}{\partial s}$ converges to 0 in any compact subdomain of $(-\infty, \infty) \times [0,1] \times M$. Since $\phi^{(l)}$ satisfies uniform $C^{1,1}$ bound, then $\phi^{(l)}$ converges uniformly to $\bar{\varphi}$ in the $C^{1,\alpha}$ norm. Using uniqueness for the complex Monge Ampere equation, we can show that $\bar{\varphi}$ is the unique $C^{1,1}$ geodesic obtained in [9]. $\square$

Now we are ready to prove[10]

Lemma 4.3. *For any two Kähler metrics $\varphi_0, \varphi_1 \in \mathcal{H}$ with $\varphi(t, \cdot)$ being the unique $C^{1,1}$ geodesic connecting these two metrics such that $\varphi(0,x) = \varphi_0$ and $\varphi(1,x) = \varphi_1$, we have*

$$\left(d\mathbf{E} \mid_{\varphi_0}, \frac{\partial \varphi}{\partial t} \mid_{t=0} \right) \leq \left(d\mathbf{E} \mid_{\varphi_1}, \frac{\partial \varphi}{\partial t} \mid_{t=1} \right).$$

Remark 4.4. *Even though the K energy is well defined along any $C^{1,1}$ geodesic path, its derivative is in general not well defined. Thus, the evaluation of the K energy form on $\frac{\partial \varphi}{\partial t} \mid_{t=1}$ is always bigger than it evaluation on $\frac{\partial \varphi}{\partial t} \mid_{t=0}$.*

Proof. Let $\kappa : (-\infty, \infty) \to \mathbf{R}$ be a smooth non-negative function such that $\kappa \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and vanishes outside of $[-\frac{3}{4}, \frac{3}{4}]$. Set

$$\kappa^{(l)}(s) = \frac{1}{v} \kappa(\frac{s}{l}), \qquad \text{where } v = \int_{-\infty}^{\infty} \kappa(s)\, d\, s.$$

Set

$$f^{(l)}(t) = \int_{-\infty}^{\infty} \kappa^{(l)}(s) \frac{d\,\mathbf{E}}{d\,t}(s,t)\, d\, s.$$

[10]This lemma should be considered as a natural extension of chen-tian [12].

and for any integer $m < l$, we can also define

$$f^{(ml)}(t) = \int_{-\infty}^{\infty} \kappa^{(m)}(s) \frac{d\,\mathbf{E}^{(l)}}{d\,t}(s,t)\; d\,s.$$

Then

$$\begin{aligned} f^{(ml)}(0) &= \int_{-\infty}^{\infty} \kappa^{(m)}(s) \frac{d\,\mathbf{E}^{(l)}}{d\,t} \mid_{(s,0)}\; d\,s \\ f^{(ml)}(1) &= \int_{-\infty}^{\infty} \kappa^{(m)}(s) \frac{d\,\mathbf{E}^{(l)}}{d\,t} \mid_{(s,1)}\; d\,s \end{aligned}$$

Now

$$\begin{aligned} f^{(ml)}(1) - f^{(ml)}(0) &= \int_0^1 \frac{df^{(ml)}}{d\,t}\; d\,t \\ &= \int_0^1 \int_{-\infty}^{\infty} \kappa^{(m)}(s)\; \frac{\partial^2 \mathbf{E}^{(l)}}{\partial\, t^2}\; d\,s\; d\,t \\ &= \int_0^1 \int_{-\infty}^{\infty} \kappa^{(m)}(s)\; \triangle_{s,t} \mathbf{E}^{(l)}\; d\,s\; d\,t - \int_0^1 \int_{-\infty}^{\infty} \kappa^{(m)}(s)\; \frac{\partial^2 \mathbf{E}^{(l)}}{\partial\, s^2}\; d\,s\; d\,t \\ &\geq -\int_0^1 \int_{-\infty}^{\infty} \kappa^{(m)}(s)\; \frac{\partial^2 \mathbf{E}^{(l)}}{\partial\, s^2}\; d\,s\; d\,t \\ &= -\int_0^1 \int_{-\infty}^{\infty} \frac{d^2 \kappa^{(m)}(s)}{d\,s^2}\; \mathbf{E}^{(l)}(s,t)\; d\,s\; d\,t \\ &= -\tfrac{1}{m^2}\tfrac{1}{v} \int_0^1 \int_{-\infty}^{\infty} \frac{d^2 \kappa^{(m)}}{d\,s^2} \mid_{\frac{s}{m}}\; \mathbf{E}^{(l)}(s,t)\; d\,s\; d\,t \end{aligned}$$

Note that $|\mathbf{E}^{(l)}(s,t)|$ has a unform bound C. Thus,

$$\begin{aligned} \tfrac{1}{m^2}\tfrac{1}{v} \mid \int_{-\infty}^{\infty} \frac{d^2 \kappa^{(m)}}{d\,s^2} \mid_{\frac{s}{m}}\; \mathbf{E}^{(l)}(s,t)\; d\,s \mid\; &\leq \tfrac{1}{m^2}\tfrac{1}{v} \int_{-\infty}^{\infty} \mid \frac{d^2 \kappa^{(m)}}{d\,s^2} \mid_{\frac{s}{m}} \mid \mathbf{E}^{(l)}(s,t)\; d\,s \\ &\leq \tfrac{C}{v\; m^2} \int_{-\infty}^{\infty} \mid \frac{d^2 \kappa^{(m)}}{d\,s^2} \mid_{\frac{s}{m}} \mid\; d\,s \\ &= \tfrac{C}{v\; m} \int_{-\infty}^{\infty} \mid \frac{d^2 \kappa^{(m)}}{d\,s^2} \mid_{s} \mid\; d\,s = \frac{C}{v\; m} \int_{-1}^{1} \mid \frac{d^2 \kappa^{(m)}}{d\,s^2} \mid_{s} \mid\; d\,s \\ &\leq \tfrac{C}{m} \end{aligned}$$

for some uniform constant C. Therefore, we have

$$\begin{aligned} f^{(ml)}(1) - f^{(ml)}(0) &\geq -\frac{1}{m^2}\frac{1}{v} \int_0^1 \int_{-\infty}^{\infty} \frac{d^2\kappa}{d\,s^2} \mid_{\frac{s}{m}}\; \mathbf{E}^{(l)}(s,t)\; d\,s\; d\,t \\ &\geq -\int_0^1 \frac{1}{m^2}\frac{1}{v} \mid \int_{-\infty}^{\infty} \frac{d^2\kappa}{d\,s^2} \mid_{\frac{s}{m}}\; \mathbf{E}^{(l)}(s,t)\; d\,s \mid\; d\,t \\ &\geq -\int_0^1 \frac{C}{m}\; d\,t = -\frac{C}{2m}. \qquad (4.4) \end{aligned}$$

Recall that

$$\begin{aligned} f^{(ml)}(1) &= \int_{-\infty}^{\infty} \kappa^{(m)}(s) \frac{d\,\mathbf{E}^{(l)}}{d\,t}(s,1)\,d\,s \\ &= \int_{-\infty}^{\infty} \kappa^{(m)}(s) \int_M (\underline{R} - R(\varphi_1)) \frac{\partial \phi^{(l)}}{\partial t} \omega_{\varphi_1}^n \,d\,s. \end{aligned}$$

For any fixed m, by Lemma 4.2, $\phi^{(l)}$ uniformly converges to φ in $\Sigma^{(m)} \times M$. In particular, $\frac{\partial \phi^{(l)}}{\partial t}$ converges strongly in C^α norm to $\frac{\partial \varphi}{\partial t}$ in $\Sigma^{(m)} \times M$. Thus, fixing m and letting $l \to \infty$, we have

$$\begin{aligned} \lim_{l\to\infty} f^{(ml)}(1) &= \int_{-\infty}^{\infty} \kappa^{(m)}(s) \int_M (\underline{R} - R(\varphi_1)) \frac{\partial \varphi}{\partial t} \mid_{t=1} \;\omega_{\varphi_1}^n \,d\,s \\ &= \lim_{l\to\infty} \int_{-\infty}^{\infty} \kappa^{(m)}(s) \int_M (\underline{R} - R(\varphi_1)) \frac{\partial \phi^{(l)}}{\partial t} \mid_{t=1} \omega_{\varphi_1}^n \,d\,s \\ &= \int_{-\infty}^{\infty} \kappa^{(m)}(s) \int_M (\underline{R} - R(\varphi_1)) \frac{\partial \varphi}{\partial t} \mid_{t=1} \omega_{\varphi_1}^n \,d\,s \\ &= \int_M (\underline{R} - R(\varphi_1)) \frac{\partial \varphi}{\partial t} \mid_{t=1} \omega_{\varphi_1}^n. \end{aligned}$$

Similarly, we can prove

$$\lim_{l\to\infty} f^{(ml)}(0) = \int_M (\underline{R} - R(\varphi_1)) \frac{\partial \varphi}{\partial t} \mid_{t=0} \omega_{\varphi_0}^n.$$

Plugging this into inequality 4.4, we have

$$(d\,\mathbf{E}, \frac{\partial \varphi}{\partial t} \mid_{t=1})_{\varphi_1} - (d\,\mathbf{E}, \frac{\partial \varphi}{\partial t} \mid_{t=0})_{\varphi_0} \geq -\frac{C}{m}.$$

As $m \to \infty$, we have

$$(d\,\mathbf{E}, \frac{\partial \varphi}{\partial t} \mid_{t=1})_{\varphi_1} \geq (d\,\mathbf{E}, \frac{\partial \varphi}{\partial t} \mid_{t=0})_{\varphi_0}.$$

The Lemma is then proved. $\square$

4.3 The lower bound of the Calabi energy

Note that the first derivative of the K energy functional is always non-decreasing along a geodesic ray. Thus, the ¥ invariant is always well defined along any relative $C^{1,1}$ geodesic ray. Now we are ready to prove Theorem 1.6.

Proof. Suppose $\rho : [0, \infty) \to \infty$ is a geodesic ray parametrized by arc length such that

$$\lim_{t\to\infty} (d\,\mathbf{E}, \frac{\partial \rho}{\partial t})_{\rho(t)} < -\infty.$$

For any Kähler potential $\varphi_0 \in \mathcal{H}$, consider the unique $C^{1,1}$ geodesic connecting φ_0 to $\rho(l)$. Represent this geodesic by $\psi_l : [0, L_l] \to \mathcal{H}$ and parametrize by arc length. Applying the preceding Theorem, we have

$$\begin{array}{ll} & (d\,\mathbf{E}, \frac{\partial \psi_l}{\partial s} \mid_{s=0})_{\varphi_0} \\ \leq & (d\,\mathbf{E}, \frac{\partial \psi_l}{\partial s} \mid_{s=L_l})_{\rho(l)} \\ = & (d\,\mathbf{E}, \frac{\partial \psi_l}{\partial s} \mid_{s=L_l} - \frac{\partial \rho}{\partial t} \mid_{t=l})_{\rho(l)} + (d\,\mathbf{E}, \frac{\partial \rho}{\partial t} \mid_{t=l})_{\rho(l)} \\ \leq & \left(\int_M (R(\rho(l)) - \underline{R})^2 \omega_\rho^n\right)^{\frac{1}{2}} \cdot \left(\int_M (\frac{\partial \rho}{\partial t} \mid_{s=L_l} - \frac{\partial \psi_l}{\partial t} \mid_{s=L_l})^2 \omega_{\rho(l)}^n\right)^{\frac{1}{2}} + (d\,\mathbf{E}, \frac{\partial \rho}{\partial t} \mid_{t=l})_{\rho(l)} \\ \leq & C(1 - (\frac{\partial \rho}{\partial t} \mid_{t=l}, \ \frac{\partial \psi_l}{\partial s}_{s=L_l})_{\rho(l)})^{\frac{1}{2}} + (d\,\mathbf{E}, \frac{\partial \rho}{\partial t} \mid_{t=l})_{\rho(l)} \end{array}$$

Since $\mathcal{H}$ is a non-positively curved manifold in the sense of Alexander, we have

$$(\frac{\partial \rho}{\partial t} \mid_l, \ \frac{\partial \psi_l}{\partial s}_{s=L_l})_{\rho(l)} \to 1$$

as $l \to \infty$. Then, we have

$$\begin{array}{lll} ¥(\rho) & = & \liminf_{l\to\infty} (d\,\mathbf{E}, \frac{\partial \rho}{\partial t} \mid_l)_{\rho(l)} \leq (d\,\mathbf{E}, \frac{\partial \psi_l}{\partial s} \mid_{s=0})_{\varphi_0} \\ & \leq & \left(\int_M (R(\omega_{\varphi_0}) - \bar{R})^2 \omega_{\varphi_0}^n\right)^{\frac{1}{2}} \left(\int_M (\frac{\partial \psi_l}{\partial s} \mid_{s=0})^2 \ \omega_{\varphi_0}^n\right)^{\frac{1}{2}} \\ & = & (Ca(\omega_{\varphi_0}))^{\frac{1}{2}}. \end{array}$$

In other words, we have

$$Ca(\omega_{\varphi_0}) \geq ¥(\rho)^2.$$

Our theorem follows from here directly! □

Now we are ready to prove Theorem 1.5.

Proof. Let $\mathcal{X}_c$ to be the *a priori* extremal vector field. Suppose $g \in \mathcal{H}_K$. Suppose that $\omega_{\rho(t)} (t \in (-\infty, \infty))$ is the one paramter family of Kähler metrics generated by pulling the Kähler metrics ω_g in the direction of $Re(X)$. It is straightforward to check that $\rho(t)$ satisfies the geodesic equation and

$$\frac{d\,E(\omega_{\rho(t)})}{d\,t} = \pm \mathcal{F}_{\mathcal{X}_c}([\omega]).$$

Select one direction so that

$$\frac{d\,E(\omega_{\rho(t)})}{d\,t} = -\mathcal{F}_{\mathcal{X}_c}([\omega]).$$

Note that the length element of this geodesic line is

$$\begin{array}{lll} \int_M \left(\frac{\partial \rho}{\partial t}\right)^2 \omega_{\rho(t)}^n & = & (\theta_{\mathcal{X}_c}, \theta_{\mathcal{X}_c}) = -(\theta_{cX_c}, R(\rho) - \bar{R}) \\ & = & -\int_M \frac{\partial \rho}{\partial t} (R(\rho(t)) - \bar{R}) \omega_{\rho(t)}^n = \mathcal{F}_{\mathcal{X}_c}. \end{array}$$

Now, if we re-paramterize by arc length, then the ¥ invariant along this geodesic line satisies

$$¥(\rho)^2 = \mathcal{F}_{\mathcal{X}_c}([\omega]).$$

Our theorem then follows from Theorem 1.6.

□

5 On the lower bound of the geodesic distance

Let us prove Theorem 1.2 first.

Proof. We follow the notations of Subsection 5.2. Set

$$E^{(ml)}(t) = \int_{-\infty}^{\infty} k^{(m)}(s)\; E^{(l)}(s,t)\; d\,s, \qquad \forall m \le l \in \mathbb{N}.$$

Then,

$$E^{(ml)}(0) = E(\varphi_0),\; E^{(ml)}(1) = E(\varphi_1)$$

and

$$\frac{d\,E^{(ml)}}{d\,t}(t) = f^{(ml)}(t), \qquad \forall\, t \in [0,1].$$

Following the same calculation as in subsection 5.2, for any $0 \le t_1 < t_2 \le 1$

$$\begin{aligned} f^{(ml)}(t_2) - f^{(ml)}(t_1) &\ge -\frac{1}{m^2}\frac{1}{v}\int_{t_1}^{t_2}\int_{-\infty}^{\infty} \frac{d^2\kappa}{d\,s^2}\,|_{\frac{s}{m}}\;\; \mathbf{E}^{(l)}(s,t)\; d\,s\; d\,t \\ &\ge -\int_{t_1}^{t_2} \frac{1}{m^2}\frac{1}{v}\; \left| \int_{-\infty}^{\infty} \frac{d^2\kappa}{d\,s^2}\,|_{\frac{s}{m}}\;\; \mathbf{E}^{(l)}(s,t)\; d\,s \right| \; d\,t \\ &\ge -\int_{t_1}^{t_2} \frac{C}{m}\; d\,t = -\frac{C}{2m}. \end{aligned} \tag{5.1}$$

Thus,

$$f^{(ml)}(0) - \frac{C}{2m} \le f^{(ml)}(t) \le f^{(ml)}(1) + \frac{C}{2m}$$

Therefore,

$$\begin{aligned} E(\varphi_1) - E(\varphi_0) &= E^{(ml)}(1) - E^{(ml)}(0) \\ &= \int_0^1 \frac{d\,E^{(ml)}}{d\,t}(t)\; d\,t = \int_0^1 f^{(ml)}(t)\; d\,t \\ &\le \int_0^1 \left(f^{(ml)}(1) + \frac{C}{2m}\right)\; d\,t \\ &= \int_{-\infty}^{\infty} \kappa^{(m)}(s) \int_M (\underline{R} - R(\varphi_1))\frac{\partial \phi^{(l)}}{\partial t}\omega_{\varphi_1}^n\; d\,s + \frac{C}{2m} \end{aligned}$$

As before, let $l \to \infty$, so $\phi^{(l)}(s,t)$ converges to the geodesic $\varphi(t)$ strongly in $C^{1,\alpha}$ norm. Then, letting $m \to \infty$, we have

$$\begin{aligned} E(\varphi_1) - E(\varphi_0) &\le \int_M (\underline{R} - R(\varphi_1))\frac{\partial\varphi}{\partial t}\omega_{\varphi_1}^n \\ &\le \sqrt{Ca(\varphi_1)}\sqrt{\int_M \left(\frac{\partial\varphi}{\partial t}\right)^2 \omega_{\varphi_1}^n} \\ &= \sqrt{Ca(\varphi_1)} \cdot d(\varphi_0, \varphi_1). \end{aligned}$$

□

Corollary 1.3 follows from this theorem since the $|\varphi|_\infty$ bound will imply the geodesic distance of φ to a fixed Kähler potential is bounded.

Before we prove Theorem 1.4, we need to prove a proposition first.

Proposition 5.1. *[11] Let $Ric(\omega_\varphi) \geq -C_1$ then there is a uniform constant C such that :*

$$\inf_M \log \frac{\omega_\varphi^n}{\omega^n} \geq -4C \exp\left(2 + 2\int_M \log \frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n\right).$$

If $C_1 = 0$, then C is dimensional constant. Otherwise, C depends on C_1 and $|\varphi|_{L^\infty}$ or $\sup \varphi_- + \int_M \varphi_+ \omega^n$.

Proof. Set

$$F = \log \frac{\omega_\varphi^n}{\omega^n}.$$

The lower Ricci curvature bound implies that

$$Ric(\omega) - i\partial\bar{\partial}F \geq -C_1\omega_\varphi.$$

Taking the trace of both sides, we have

$$\triangle(F - C_1\varphi) \leq C_2$$

for some constant C_2.

Choose a constant c such that

$$\int_M \log \frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n \leq c,$$

In a fixed Kähler class, we have

$$\int_M \omega^n = Vol(M) = 1.$$

Put ϵ to be $\exp(-2-2c)$. Observe that

$$\log \frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n \geq -e^{-1}\omega^n,$$

We have

$$\begin{aligned} c \quad & \geq \left(\int_{\epsilon\omega_\varphi^n > \omega^n} + \int_{\epsilon\omega_\varphi^n \leq \omega^n}\right)\left(\log \frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n\right) \\ & \geq \int_{\epsilon\omega_\varphi^n > \omega^n}\left(\log \frac{1}{\epsilon}\right)\omega_\varphi^n + \int_{\epsilon\omega_\varphi^n \leq \omega^n}\left(-e^{-1}\omega^n\right) \\ & > 2(1+c)\int_{\epsilon\omega_\varphi^n > \omega^n}\omega_\varphi^n - 1. \end{aligned}$$

It follows that

$$\int_{\epsilon\omega_\varphi^n > \omega^n} < \frac{1}{2},$$

and consequently,

$$\int_{\omega^n \le 4\omega_\varphi^n} \omega^n \ge \epsilon \int_{\frac{\epsilon}{4}\omega^n \le \epsilon\omega_\varphi^n \le \omega^n} \omega_\varphi^n \ge \frac{1}{4},$$

because we know

$$\int_{\omega^n \le 4\omega_\varphi^n} \omega_\varphi^n > \frac{3}{4}$$

and

$$\int_{\omega^n \le \epsilon\omega_\varphi^n} > \frac{1}{2}.$$

Now by Green's formula, we have

$$(F - C_1\varphi)(p) = -\int_M G(p,q)\triangle(F - C_1\varphi)\omega^n(q) + \int_M (F - C_1\varphi)\omega^n,$$

where $G(p,q) \ge 0$ is a Green function of ω. If either $|\varphi|_{L^\infty}$ is bounded, or

$$\sup \varphi_- \le C, \qquad \text{and} \quad \int_M \varphi_+ \, \omega^n \le C,$$

then

$$\begin{aligned}
\inf_M F &\ge \inf_M F \int_{\omega^n \ge 4\omega_\varphi^n} \omega^n + \int_{\omega^n < 4\omega_\varphi^n} F\omega^n - C_1 \sup \varphi + C_1 \int_M (-\varphi_-)\omega^n \\
&\ge \int_M F\omega^n - C \\
&\ge \inf_M F \int_{\omega^n \ge 4\omega_\varphi^n} \omega^n + \int_{\omega^n < 4\omega_\varphi^n} F\omega^n - C \\
&\ge \inf_M F \int_{\omega^n \ge 4\omega_\varphi^n} \omega^n - \log 4 \int_{\omega^n < 4\omega_\varphi^n} \omega^n - C \\
&\ge \left(1 - \frac{\epsilon}{4}\right) \inf_M F - C,
\end{aligned}$$

where we can assume $\inf_M F < 0$. Therefore, we have

$$\inf_M F \ge -4C \exp(2 + 2c).$$

By the way we choose the constant c in the beginning of the proof, proposition is proved. $\square$

One more lemma is needed.

Lemma 5.2. *Let $\omega' \in [\omega]$, and suppose (M, ω') has a bounded Soblev constant. For any $\psi \in C^\infty(M)$, and $\omega' + i\partial\bar{\partial}\psi > 0$, then $\psi_+ = \max(\psi, 0)$ is uniformly bounded if $\int_M \psi_+^p \omega'^n < C$ for any $p > 1$.*

The proof is well known to the experts and we will include it here for the convenience of readers.

Proof. Without loss of generality, may assume $\psi \geq 1$ for simplicity. We start from

$$n + \triangle'\psi > 0,$$

here $\triangle'$ is the Laplacian operator of ω'. For any $p \geq 1$, we have

$$\begin{array}{rcl} \int_M n \cdot \psi^p \omega'^n & \geq & p \int_M |\nabla \psi|^2 \psi^{p-1} \omega'^n \\ & = & p \int_M |\nabla \psi^{\frac{p+1}{2}}|^2 \psi^{p-1} \omega'^n \\ & = & \frac{4p}{(p+1)^2} \int_M |\nabla \psi^{\frac{p+1}{2}}|^2 \omega'^n. \end{array}$$

Since ω' has a uniform Soblev constant, we have

$$\begin{array}{rcl} c_{Sob} \left(\int_M \psi^{\frac{p+1}{2} \cdot \frac{2m}{m-2}} \right)^{\frac{m-2}{m}} & \leq & \int_M |\nabla \psi^{\frac{p+1}{2}}|^2 + (\psi^{\frac{p+1}{2}})^2 \\ & \leq & \frac{(p+1)^2}{4p} n \int_M \psi^p \omega'^n + \int_M \psi^{p+1} \omega' n \end{array}$$

Thus, there is a uniform constant C which is independent of p such that

$$\left(\int_M \psi^{(p+1) \cdot \frac{m}{m-2}} \right)^{\frac{m-2}{m}} \leq C(p+1) \int_M \psi^{p+1} \omega'^n.$$

Set

$$p_1 = q > 0, p_2 = p_1 \frac{m}{m-2}, \cdots p_{j+1} = p_j \cdot \frac{m}{m-2}, \cdots$$

Then,

$$\|\psi\|_{p_{j+1}} \leq C^{\frac{1}{p_j}} p_j^{\frac{1}{p_j}} \|\psi\|_{p_j}, \qquad , \forall j = 1, 2, \cdots.$$

In other words,

$$\|\psi\|_{p_{j+1}} \leq \|\psi\|_{p_1} \cdot C^{\sum_{k=1}^{j} \left(\frac{1}{p_j} + 1 \;\; p_j \log p_j \right)} \|\psi\|_{p_1}.$$

Let $p_1 = 1$, and $j \to \infty$, then

$$\|\psi\|_{L^\infty} \leq C \|\psi\|_{L^2}.$$

□

Now we are ready to prove Theorem 1.4.

Proof. Suppose $\varphi \in \mathcal{H}$ is a Kähler potential such that

1. $Ric(\omega_\varphi)$ is bounded from below;

2. The diameter of (M, ω_φ) is bounded from above;

3. the geodesic distance $d(0, \varphi)$ is bounded from above.

The first two conditions implies that there is a uniform Soblev and Poincare constants for (M, ω_φ). In the proof here, "C" represents a generic constant.

Now normalize φ by a constant necessary so we have

$$I(\varphi) = 0.$$

According to a theorem in [9], we have

$$\max \left(\int_M \varphi_- \omega^n, \int_M \varphi_+ \omega_\varphi^n, \right) \leq d(0, \varphi) \leq C. \tag{5.2}$$

Here φ_+, φ_- are positive and negative part of φ respectively.

Set the K energy of ω_0 being 0. Theorem 1.2 implies that

$$\mathbf{E}(\varphi) \leq \mathbf{E}(0) + \sqrt{C a(\varphi)} d(0, \varphi) \leq C.$$

If the K energy functional is quasi-proper (c.f., the detailed expression of the K energy functional [10]), we obtain

$$\int_M \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n \leq C.$$

According G. Tian, there is a positive constant $\alpha > 0$ which depends only on the polarization $(M, [\omega])$ such that for any $\varphi \in \mathcal{H}$, we have

$$\int_M e^{-\alpha(\varphi - \sup \varphi)} \omega^n \leq C.$$

Or

$$\int_M e^{-\alpha(\varphi - \sup \varphi) - \log \frac{\omega_\varphi^n}{\omega^n}} \omega_\varphi^n \leq C.$$

Consequently, we have

$$\int_M -\alpha(\varphi - \sup \varphi) - \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n \leq C.$$

Therefore,

$$\begin{array}{rcl} \alpha \sup \varphi & \leq & \alpha \int_M \varphi \omega_\varphi^n + \int_M \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n \\ & \leq & \alpha \int_M \varphi \omega_\varphi^n + \int_M \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n \\ & \leq & C. \end{array}$$

it follows,

$$\int_M |\varphi| \omega^n \leq C. \tag{5.3}$$

By the detailed expreesion of $I(\varphi)$ (cf. equation 2.6), we have

$$J(\varphi) \leq C. \tag{5.4}$$

Alternatively, when the K energy is proper, we can obtain estimate 5.3 and 5.4 as well. Note that if the K energy functional $\mathbf{E}$ is proper in $\mathcal{H}$, we have

$$0 \leq J(\varphi) \leq C. \tag{5.5}$$

Again, from the detailed expression of $I(\varphi)$ (cf. equation 2.6), we have

$$|\int_M \varphi\omega^n| \leq C.$$

Comparing to estimate 5.2, we have

$$\int_M \varphi_+\omega^n \leq C \tag{5.6}$$

and

$$\int_M \varphi_-\omega_\varphi^n \leq C. \tag{5.7}$$

or

$$\int_M |\varphi|\omega^n \leq C. \tag{5.8}$$

Since $Ric(\omega_\varphi) \geq -C$ and the diameter is bounded from below, we have a uniform Poincare constant for (M, ω_φ). Using the Poincare inequality, we have

$$\int_M \varphi^2\omega^n + \int_M \varphi^2\omega_\varphi^n \leq C\left(J(\varphi) + \left(\int_M |\varphi|\omega_\varphi^n\right)^2\right) \leq C.$$

Recall that

$$n + \triangle\varphi \geq 0.$$

Using Moser iteration, and the J functional bound 5.4, we obtain

$$0 \leq \varphi_+ \leq C.$$

Recall that

$$n + \triangle_\varphi(-\varphi) \geq 0.$$

By the assumption that the Soblev constant of (M, ω_φ) is bounded and the L^2 norm is bounded above, appealing to Lemma 5.2, gives us

$$0 \leq \varphi_- \leq C.$$

In other words, we have

$$|\varphi|_{L^\infty} \leq C.$$

To derive an upper bound on the volume form, first note that $Ric(\omega_\varphi)$ is bounded from above, thus

$$\triangle \log \frac{\omega_\varphi^n}{\omega^n} + C_2\varphi \geq -C \tag{5.9}$$

for some constant C_2, C. Thus

$$\begin{array}{rl} & \left(\log \frac{\omega_\varphi^n}{\omega^n} + C_2\varphi\right)(x) \\ = & -\int_M G(x,y)\left(\log \frac{\omega_\varphi^n}{\omega^n} + C_2\varphi\right)\omega^n + \int_M \left(\log \frac{\omega_\varphi^n}{\omega^n} + C_2\varphi\right)\omega^n \\ \leq & +C + \log \int_M \frac{\omega_\varphi^n}{\omega^n}\omega^n + \int_M \varphi\omega^n \\ \leq & C + \int_M \varphi\omega^n. \end{array}$$

Using the fact that $|\varphi|_{L^\infty}$ is bounded, we have

$$\log \frac{\omega_\varphi^n}{\omega^n} \leq C$$

for some constant C.

To prove the metric is equivalent, we follow Yau's proof of the Calabi conjecture (cf. [13]). Folllowing notations in Subsection 3.2.2, set

$$u = \exp(-\lambda\varphi)(n + \triangle\varphi), \qquad \text{and } \ F = \log \frac{\omega_\varphi^n}{\omega^n}.$$

At the maximal point p of the function u, similar to inequality 3.15, we have

$$\triangle_\varphi \left\{\exp(-\lambda\varphi)(n+\triangle\varphi)\right\}(p) \leq 0.$$

At the point p, we have

$$\begin{array}{rcl} 0 & \geq & \triangle F - n^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}} - \lambda n(n+\triangle\varphi) \\ & & + \left(\lambda + \inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\exp\left\{\dfrac{-F}{n-1}\right\}(n+\triangle\varphi)^{\frac{n}{n-1}}. \end{array}$$

Applying inequality 5.9, we have

$$\begin{array}{rcl} 0 & \geq & -C\triangle\varphi - C_2 - n^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}} - \lambda n(n+\triangle\varphi) \\ & & + \left(\lambda + \inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\exp\left\{\dfrac{-F}{n-1}\right\}(n+\triangle\varphi)^{\frac{n}{n-1}}. \end{array}$$

Since we already have control of F from both above and below here, we can choose λ large enough, to imply that $(n+\triangle\varphi)(p)$ is uniformly bounded from above. Therefore,

$$u = \exp(-\lambda\varphi)(n + \triangle\varphi), \qquad \text{and } \ F = \log \frac{\omega_\varphi^n}{\omega^n}$$

is uniformly bounded from above (since $|\varphi|_{L^\infty}|$ is uniformly bounded). Thus

$$0 < n + \triangle\varphi \leq C$$

Thus, ω_φ is uniformly equivalent to ω. Consequently, $|\triangle F|$ is uniformly bounded. Thus, the metric is uniformly $C^{1,\alpha}$ bounded for any $\alpha \in (0,1)$.

$\square$

References

[1] C., Arezzo and G. Tian. Infinite geodesic rays in the space of Khler potentials. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2 (2003), no. 4, 617–630.

[2] E.D. Bedford and T.A. Taylor. The Drichelet problem for the complex Monge-Ampere operator. *Invent. Math.*, 37:1–44, 1976.

[3] B. Weinkove and J. Song. On the convergence and singularities of the J-flow with applications to the Mabuchi energy math.DG/0410418.

[4] Ben Weinkove On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy math.DG/0309404.

[5] L. Caffarelli, L. Nirenberg and J. Spruck. The dirichlet problem for nonlinear second-order elliptic equation I. monge-ampere equation. *Comm. on pure and appl. math.*, XXXVII:369–402, 1984.

[6] E. Calabi. Extremal Kähler metrics. In *Seminar on Differential Geometry*, volume 16 of *102*, pages 259–290. Ann. of Math. Studies, University Press, 1982.

[7] E. Calabi. Extremal Kähler metrics, II. In *Differential geometry and Complex analysis*, pages 96–114. Springer, 1985.

[8] E. Calabi and X. X. Chen. Space of Kähler metrics and Calabi flow, 2002. Journal of Differential Geometry.

[9] X. X. Chen. Space of Kähler metrics. *Journal of Differential Geometry*, 56(2):189–234, 2000.

[10] X. X. Chen. On the lower bound of the Mabuchi energy and its application. *Internat. Math. Res. Notices* 2000, no. 12, 607–623.

[11] X. X. Chen, G. Tian, *Ricci flow on Kähler-Einstein surfaces*, Invent. math. **147** (2002), 487-544

[12] X. X. Chen and G. Tian. Foliation by holomorphic discs and its application in Kähler geometry, 2003. submitted

[13] X. X. Chen and W. Y. He On the Calabi flow, Preprint, 2005.

[14] R.R. Coifman and S. Semmes. Interpolation of Banach spaces, Perron process, and Yang Mills. *Amer. J. Math.*, 115(2):243–278, 1993.

[15] S. K. Donaldson. Symmetric spaces, Kähler geometry and Hamiltonian dynamics. *Amer. Math. Soc. Transl. Ser. 2, 196*, pages 13–33, 1999. Northern California Symplectic Geometry Seminar.

[16] S. K. Donaldson. Holomorphic discs and the complex Monge-Ampère equation, 2001. to appear in Journal of Sympletic Geometry.

[17] S. K. Donaldson. Scalar curvature and projective embeddings. II. *Q. J. Math.* 56 (2005), no. 3, 345–356.

[18] S. K. Donaldson. Interior estimates for solutions of Abreu's equation. *Collect. Math.* 56 (2005), no. 2, 103–142.

[19] S. K. Donaldson Lower bounds on the Calabi functional. math.DG/0506501.

[20] A. Futaki An obstruction to the existence of Einstein Kähler metrics. *Invent. Math.* 73 (1983), no. 3, 437–443.

[21] A. Futaki and T. Mabuchi. Bilinear forms and extremal Khler vector fields associated with Khler classes. *Math. Ann.* 301 (1995), no. 2, 199–210.

[22] B. Guan. The Dirichlet problem for complex Monge-Ampere equations and regularity of the plui-complex green function. *Comm. Ana. Geom.*, 6(4):687–703, 1998.

[23] A. D. Hwang. On the Calabi energy of Extremal Kähler metrics. *International Journal of Mathematics*, 6(6):825–830, 1995.

[24] J.T. Kohn L. Nirenberg L. Caffarelli and J. Spruck. The dirichlet problem for nonlinear second-order elliptic equation II. complex monge-ampere equation. *Comm. on pure and appl. math.*, 38:209–252, 1985.

[25] T. Mabuchi. Some symplectic geometry on compact Kähler manifolds I. *Osaka, J. Math.*, 24:227–252, 1987.

[26] T. Mabuchi. An energy-theoretic approach to the Hitchin-Kobayashi correspondence for manifolds. I. *Invent. Math.* 159 (2005), no. 2, 225–243.

[27] S. Paul and G.Tian. Analysis of geometric stability. *Int. Math. Res. Not.* 2004, no. 48, 2555–2591.

[28] S. Paul. Geometric analysis of Chow Mumford stability. *Adv. Math.* 182 (2004), no. 2, 333–356.

[29] D.H. Phong, J. Sturm The Monge-Ampère operator and geodesics in the space of Kähler potentials. math.DG/0504157.

[30] W.D. Ruan. On the convergence and collapsing of Kähler metrics. *J. Diff. Geom.*, 52:1-40, 1999.

[31] S. Semmes. Complex Monge-Ampère equations and sympletic manifolds. *Amer. J. Math.*, 114:495–550, 1992.

[32] G. Tian. Khler-Einstein metrics with positive scalar curvature. *Invent. Math.* 130 (1997), no. 1, 1–37.

[33] S. T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampere equation, I^*. *Comm. Pure Appl. Math.*,, 31:339–441, 1978.